C. SCHUR-ZASSENHAUS, THE ABELIAN CASE

The aim of the next two worksheets will be to prove:

**Theorem Schur-Zassenhaus:** Let \(1 \to A \to B \to C \to 1\) be a short exact sequence of finite groups where \(\gcd(\#(A), \#(C)) = 1\). Then this sequence is right split, so \(B \cong A \rtimes C\).

This is the start of an answer to the question “how are groups assembled out of smaller groups”: When you put groups of relatively prime order together, you just get semidirect products.

Today, we’ll be proving the case where \(A\) is abelian. Here is our main result:

**Theorem:** Let \(A\) be an abelian group, \(C\) a finite group of size \(n\), and suppose that \(a \mapsto a^n\) is a bijection from \(A\) to \(A\). Let \(1 \to A \to B \to C \to 1\) be a short exact sequence. Then this sequence is right split.

**Problem C.1.** Show that, if \(A\) is a finite abelian group and \(n\) an integer such that \(\gcd(\#(A), n) = 1\), then \(a \mapsto a^n\) is a bijection. Thus, the above Theorem does imply the Schur-Zassenhaus theorem for \(A\) abelian.

From now on, let \(A\) be an abelian group, let \(C\) be a finite group and let \(1 \to A \to B \xrightarrow{\beta} C \to 1\) be a short exact sequence. We abbreviate \(\#(C)\) to \(n\); we will not introduce the hypothesis on \(a \mapsto a^n\) until later. We’ll identify \(A\) with its image in \(B\).

Let \(S\) be the set of right inverses of \(\beta\), meaning maps \(\sigma : C \to B\) such that \(\beta(\sigma(c)) = c\). We emphasize that \(\sigma\) is not required to be compatible with the group multiplication in any way. Let \(B\) act on \(S\) by \((b\sigma)(c) = b\sigma(\beta(b)^{-1}c)\).

**Problem C.2.** Check that this is an action.

Let \(\sigma_1\) and \(\sigma_2\) ∈ \(S\). Set \(d(\sigma_1, \sigma_2) = \prod_{c \in C} (\sigma_1(c)\sigma_2(c)^{-1})\).

We don’t have to specify the order of the product, because every term is in \(A\).

**Problem C.3.** Show that \(d(\sigma_1, \sigma_2)d(\sigma_2, \sigma_3) = d(\sigma_1, \sigma_3)\) and \(d(\sigma_1, \sigma_2) = d(\sigma_2, \sigma_1)^{-1}\).

**Problem C.4.** For the action of \(B\) on \(S\) described above, check that \(d(b\sigma_1, b\sigma_2) = bd(\sigma_1, \sigma_2)b^{-1}\).

Define \(\sigma_1 \equiv \sigma_2\) if \(d(\sigma_1, \sigma_2) = 1\).

**Problem C.5.** Check that \(\equiv\) is an equivalence relation.

Define \(X\) to be the set of equivalence classes of \(S\) module the relation \(\equiv\).

**Problem C.6.** Check that the action of \(B\) on \(S\) descends to an action of \(B\) on \(X\).

Now, we impose the condition that \(a \mapsto a^n\) is an automorphism of \(A\).

**Problem C.7.** Show that the subgroup \(A\) of \(B\) acts on \(X\) with a single orbit and trivial stabilizers.

The following problem was on the problem sets; check that everyone knows how to do it:

**Problem C.8.** You have shown that \(B\) acts on \(X\), and that the restriction of this action to \(A\) has a single orbit and trivial stabilizers. Explain why this means that \(1 \to A \to B \to C \to 1\) is right split.

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1This approach is closely based on that of Kurzweil and Stellmacher, *The Theory of Finite Groups*, Chapter 3.3, Springer-Verlag (2004).