D. The Schur-Zassenhaus Theorem, General Case

Today’s goal is to prove:

**Theorem (Schur-Zassenhaus):** Let $A$ and $C$ be finite groups with $\gcd(|A|, |C|) = 1$. Then any short exact sequence $1 \to A \to B \to C \to 1$ is right split.

We introduce the following (not standard) terminology: We’ll say that a pair of groups $(A, C)$ is **straightforward** if every short exact sequence $1 \to A \to B \to C \to 1$ is right split. The abelian Schur-Zassenhaus theorem shows that if $A$ is abelian and $\gcd(|A|, |C|) = 1$, then $(A, C)$ is straightforward.

**Problem D.1.** Suppose that $(A_1, C)$ and $(A_2, C)$ are straightforward and there is a short exact sequence $1 \to A_1 \to A \to A_2 \to 1$ with $A_1$ canonical in $A$. Show that $(A, C)$ is straightforward. **Hint/Warning:** Unfortunately, I think this first problem is one of the hardest. First use that $(A_2, C)$ is straightforward, then use that splitting to build a new sequence which we can split using that $(A_1, C)$ is straightforward.

**Problem D.2.** Let $C$ be a finite group, let $p$ be a prime not dividing $|C|$ and let $P$ be a $p$-group. Show that $(P, C)$ is straightforward.

Let $p$ be a prime dividing $|A|$ and let $P$ be a $p$-Sylow subgroup of $A$. Let $1 \to A \to B \to C \to 1$ be a short exact sequence, with $\gcd(|A|, |C|) = 1$. **Assume inductively that we have shown** $(A', C)$ is straightforward whenever $\gcd(|A'|, |C|) = 1$ for $|A'| < |A|$.

Recall that $N_A(P) = \{a \in A : aPa^{-1} = P\}$ and likewise for $N_B(P)$.

**Problem D.3.** Show that $P$ is canonical in $N_A(P)$.

**Problem D.4.** Suppose that $A = N_A(P)$. Prove that $1 \to A \to B \to C \to 1$ is right split.

So we may now assume that $N_A(P) \neq A$.

**Problem D.5.** With $A$, $B$, $C$, $P$ as above, show that $1 \to N_A(P) \to N_B(P) \to C \to 1$ is exact.

**Problem D.6.** Show that $1 \to A \to B \to C \to 1$ is right split.