Recall:

**Theorem/Definition** Let $L/K$ be a field extension of finite degree. The following are equivalent:

1. We have $\# \text{Aut}(L/K) = [L : K]$.
2. The fixed field of $\text{Aut}(L/K)$ is $K$.
3. For every $\theta \in L$, the minimal polynomial of $\theta$ over $K$ is separable and splits in $L$.
4. $L$ is the splitting field of a separable polynomial $f(x) \in K[x]$.

A field extension $L/K$ which satisfies these equivalent definitions is called **Galois**.

Given a subfield $F$ with $K \subseteq F \subseteq L$, we write $\text{Stab}(F)$ for the subgroup of $\text{Gal}(L/K)$ fixing $F$; given a subgroup $H$ of $\text{Gal}(L/K)$, we write $\text{Fix}(H)$ for the subfield of $L$ fixed by $H$. Our next main goal will be to show:

**The fundamental Theorem of Galois theory** Let $L/K$ be a Galois extension with Galois group $G$. The maps $\text{Stab}$ and $\text{Fix}$ are inverse bijections between the set of subgroups of $G$ and the set of intermediate fields $F$ with $K \subseteq F \subseteq L$. Moreover, if $F_1 \subseteq F_2$, then $\text{Stab}(F_1) \supseteq \text{Stab}(F_2)$ and $[\text{Stab}(F_1) : \text{Stab}(F_2)] = [F_2 : F_1]$. If $H_1 \subseteq H_2$ then $\text{Fix}(H_1) \supseteq \text{Fix}(H_2)$ and $[\text{Fix}(H_1) : \text{Fix}(H_2)] = [H_2 : H_1]$.

We start by proving some basic results about $\text{Fix}$ and $\text{Stab}$.

**Problem E.1.**

1. Show that, if $F_1 \subseteq F_2$ then $\text{Stab}(F_1) \supseteq \text{Stab}(F_2)$.
2. Show that, if $H_1 \subseteq H_2$ then $\text{Fix}(H_1) \supseteq \text{Fix}(H_2)$.

**Problem E.2.**

1. Show that $\text{Stab}$(Fix($H$)) $\supseteq H$.
2. Show that Fix(Stab($F$)) $\supseteq F$.

The Fundamental Theorem tells us that both of the $\supseteq$’s in Problem E.2 are actually equality, but we don’t know that yet.

We now give examples. Here is a table of the subgroups of $S_3$:

```
S3
/|\
/ \|\
A3 ⟨(12)⟩ ⟨(13)⟩ ⟨(23)⟩
/ \|\{|e|}
```

**Problem E.3.** Let $L = \mathbb{Q}(x_1, x_2, x_3)$, let $S_3$ act on $L$ by permuting the variables and let $K = \text{Fix}(S_3)$. Describe the subfield of $L$ fixed by each of the subgroups of $S_3$.

**Problem E.4.** Let $L$ be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. We number the roots of $x^3 - 2$ as $\sqrt[3]{2}$, $\omega \sqrt[3]{2}$ and $\omega^2 \sqrt[3]{2}$, where $\omega$ is a primitive cube root of 1. Described the subfield of $L$ fixed by each of the subgroups of $S_3$.

Now we prove the theorem!

**Problem E.5.** Both parts of this problem are things you already did, your job is just to remember when you did them.

1. Let $L/K$ be a Galois extension. Let $F$ be a field with $K \subseteq F \subseteq L$. Show that $|\text{Stab}(F)| = [L : F]$.
2. Let $L/K$ be a Galois extension. Let $H$ be a subgroup of $\text{Gal}(L/F)$. Show that $[L : \text{Fix}(H)] = |H|$.

**Problem E.6.** Prove that the maps $\text{Fix}$ and $\text{Stab}$ in the Fundamental Theorem are mutually inverse.

**Problem E.7.** Check the remaining claims of the Fundamental Theorem.