See the course website for homework policies.

**Problem 1.** Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 2.

**Problem 2.** This problem is your opportunity to write up and improve Problem 6.1 from the worksheets: Let $G$ be a group and let $N$ be a subgroup of $G$. Show that the following are equivalent:

1. For all $g \in G$, we see $N = gNg^{-1}$.
2. $N$ is a union of (some of the) conjugacy classes of $G$.
3. All elements of $G/N$ have the same stabilizer, for the left action of $G$ on $G/N$.
4. For all $g \in G$, we have $gN = Ng$.
5. For all $g \in G$, there is a $g' \in G$ such that $gN = Ng'$.
6. For all $g_1$ and $g_2 \in G$, we have $g_1Ng_2N = g_1g_2N$.
7. There is a group structure on $G/N$ such that the map $g \mapsto gN$ from $G$ to $G/N$ is a group homomorphism.
8. There is a group $H$ and a group homomorphism $\alpha : G \to H$ such that $N = \operatorname{Ker}(\alpha)$.

Many of the implications are very short. That’s fine!

**Problem 3.** Write up worksheet problem 6.3, 6.4, 7.1 or 7.2.

**Problem 4.** This problem is meant to help you think about conjugacy classes: Let $G$ be a group and let $g_1$ and $g_2$ be conjugate elements of $G$.

1. Show that $g_1$ and $g_2$ have the same order.
2. For any positive integer $k$, show that $g_1$ has a $k$-th root in $G$ if and only if $g_2$ has a $k$-th root.
3. Suppose that $G$ acts on a set $X$, and let $X_i = \{x \in X : g_i(x) = x\}$. Show that there is an $h \in G$ such that $hX_1 = X_2$.

**Problem 5.** Let $G$ be a group, $N$ a normal subgroup of $G$ and $g$ an element of $N$. Let $\operatorname{Conj}_G(g)$ be the conjugacy class of $g$ in $G$. In this problem, we will discuss how $\operatorname{Conj}_G(g)$ splits into conjugacy classes of $N$.

1. Show that $\operatorname{Conj}_G(g)$ is a union of conjugacy classes of $N$.
2. Suppose that $G/N$ is finite. Let $C_G(g) = \{h \in G : gh = hg\}$. Let $\pi$ be the quotient map $G \to G/N$. Show the number of $N$-conjugacy classes in $\operatorname{Conj}_G(g)$ is $\frac{\#(G/N)}{\#(C_G(g))}$.
3. Let $n$ be an odd number. Show that $S_n$-conjugacy class of $(123\cdots n)$ splits into two conjugacy classes in $A_n$.
4. Let $p$ be an odd prime. Show that the $\operatorname{GL}_2(\mathbb{F}_p)$-conjugacy class of $[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}]$ splits into two conjugacy classes in $\operatorname{SL}_2(\mathbb{F}_p)$.

**Problem 6.** Let $G$ be a group and $H$ a subgroup. The subgroup $H$ is called canonical if $\phi(H) = H$ for every automorphism $\phi$ of $G$.

1. Show that, if $H$ is canonical in $G$, then $H$ is normal in $G$.

Let $A \subset B \subset C$ be a chain of groups. Give a proof or counterexample to each statement:

2. If $A$ is canonical in $B$ and $B$ is canonical in $C$, then $A$ is canonical in $C$.
3. If $A$ is canonical in $B$ and $B$ is normal in $C$, then $A$ is normal in $C$.
4. If $A$ is normal in $B$ and $B$ is canonical in $C$, then $A$ is normal in $C$.
5. If $A$ is normal in $B$ and $B$ is normal in $C$, then $A$ is normal in $C$. 