See the course website for homework policies.

**Problem 1.** Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 5.

**Problem 2.** Please write up two of 13.4, 13.5, 13.6, 13.7.

**Problem 3.** Describe all actions of $C_2$ on $C_{15}$ by group automorphisms.

**Problem 4.** Let $G$ be the group of all symmetries of the plane of the form $[x \ y] \mapsto (-1)^{by+ax}$ for $a, b \in \mathbb{Z}$. This is the group of symmetries of the pattern below (imagined to fill the plane).

![Pattern diagram]

(1) Show that there is a short exact sequence $1 \to \mathbb{Z}^2 \to G \to C_2 \to 1$.

(2) Show that this sequence is not right split.

**Problem 5.** Let $R$ be a ring (not assumed commutative) and let $I$ be a two sided ideal of $R$. We define $I^m$ to be the two sided ideal generated by all products $x_1x_2 \cdots x_m$ for $x_1, x_2, \ldots, x_m \in I$.

We define the ideal $N$ to be **nilpotent** if there is a positive integer $m$ such that $N^m = (0)$. Let $N$ be a nilpotent ideal and let $U$ be the group $\{1 + x : x \in N\}$. Show that $U$ is a nilpotent group.

**Problem 6.** Let $G$ be a group.

(1) The **upper (or ascending) central series** of $G$ is defined inductively as follows: $U_0 = \{e\}$ and $U_{k+1} = \pi_k^{-1}(Z(G/U_k))$, where $\pi_k$ is the projection $G \to G/U_k$. Show that $G$ is nilpotent if and only if $U_N = G$ for some $G$.

(2) The **lower (or descending) central series** is defined inductively as follows: $L^0 = G$ and $L^{k+1}$ is the group generated by all products $ghg^{-1}h^{-1}$ with $g \in G$ and $h \in L^k$. Show that $G$ is nilpotent if and only if $L^N = \{e\}$ for some $G$.

**Problem 7.** Let $p$ be an odd prime and let $G$ be a group of order $p^3$. The aim of this problem is to show that $G$ is isomorphic to one of:

$$C_p^3, \quad C_p \times C_p, \quad C_{p^3}, \quad C_p \times \langle [1 \ 0 \ 0] \rangle C_p, \quad C_{p^2} \times \langle (1+ip) \rangle C_p.$$

(1) Show that there is a central extension $1 \to Z \to G \to C \to 1$ with $Z \cong C_p$ and either $C \cong C_{p^2}$ or $C \cong C_{p^2}$.

(2) If $C \cong C_{p^2}$, show $G$ is abelian and in the list above. **From now on, we assume $C \cong C_{p^2}$**.

(3) Let $g_1$ and $g_2 \in G$ and set $z = g_1g_2g_1^{-1}g_2^{-1}$. Show that $g_1^p$, $g_2^p$ and $z$ are in $Z$.

(4) Show that $(g_1g_2)^k = g_1^k g_2^k z^{-(k^2)}$.

(5) Show that the map $g \mapsto g^p$ is a group homomorphism $G \to Z$ and that it factors through the quotient $C$. (Here is where you will need that $p$ is odd; this is false for $Q_8$.)

(6) Show that we can choose $g_1$ and $g_2$ in $G$, mapping to a basis of $C$, such that $g_1^p = 1$.

(7) Set $A' = \langle g_2 \rangle Z$ and $C' = \langle g_1 \rangle$. Show that $G = A' \rtimes C'$ and that $A'$ is either $C_{p^2}$ or $C_{p^2}$.

(8) Classify the actions of $C'$ on $A'$ and thus show that $G$ is one of the groups listed above.