See the course website for homework policies.

**Problem 1.** Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 5.

**Problem 2.** Please write up two of 13.4, 13.5, 13.6, 13.7.

**Problem 3.** Describe all actions of \( C_2 \) on \( C_{15} \).

**Problem 4.** Let \( G \) be the group of all symmetries of the plane of the form \([x \ y] \mapsto (-1)^by+a\) for \(a, b \in \mathbb{Z}\). This is the group of symmetries of the pattern below (imagined to fill the plane).

(1) Show that there is a short exact sequence \( 1 \to \mathbb{Z}^2 \to G \to C_2 \to 1 \).
(2) Show that this sequence is not right split.

**Problem 5.** Let \( R \) be a ring (not assumed commutative) and let \( I \) be a two sided ideal of \( R \). We define \( I^m \) to be the two sided ideal generated by all products \( x_1x_2\cdots x_m \) for \( x_1, x_2, \ldots, x_m \in I \).

We define the ideal \( N \) to be *nilpotent* if there is a positive integer \( m \) such that \( N^m = (0) \). Let \( N \) be a nilpotent ideal and let \( U \) be the group \( \{1 + x : x \in N\} \). Show that \( U \) is a nilpotent group.

**Problem 6.** Let \( G \) be a group.

(1) The *upper (or ascending) central series* of \( G \) is defined inductively as follows: \( U_0 = \{e\} \) and \( U_{k+1} = \pi_k^{-1}(Z(G/U_k)) \), where \( \pi_k \) is the projection \( G \to G/U_k \). Show that \( G \) is nilpotent if and only if \( U_N = G \) for some \( G \).
(2) The *lower (or descending) central series* is defined inductively as follows: \( L^0 = G \) and \( L^{k+1} \) is the group generated by all products \( ghg^{-1}h^{-1} \) with \( g \in G \) and \( h \in L^k \). Show that \( G \) is nilpotent if and only if \( L^N = \{e\} \) for some \( G \).

**Problem 7.** Let \( p \) be an odd prime and let \( G \) be a group of order \( p^3 \). The aim of this problem is to show that \( G \) is isomorphic to one of:

\[
C_p^3, \quad C_p \times C_p, \quad C_p^2, \quad C_p^2 \times C_p, \quad C_p, \quad C_p^2 \times C_p, \quad C_p^2 \times C_p, \quad C_p.
\]

(1) Show that there is a central extension \( 1 \to Z \to G \to C \to 1 \) with \( Z \cong C_p \) and either \( C \cong C_p^2 \) or \( C \cong C_p \).
(2) If \( C \cong C_p^2 \), show \( G \) is abelian and in the list above. **From now on, we assume \( C \cong C_p^2 \).**
(3) Let \( g_1 \) and \( g_2 \in G \) and set \( z = g_1g_2g_1^{-1}g_2^{-1} \). Show that \( g_1^p, g_2^p \) and \( z \) are in \( Z \).
(4) Show that \( (g_1g_2)^k = g_1^k g_2^k z^{-k} \).
(5) Show that the map \( g \mapsto g^p \) is a group homomorphism \( G \to Z \) and that it factors through the quotient \( C \). (Here is where you will need that \( p \) is odd; this is false for \( Q_8 \).)
(6) Show that we can choose \( g_1 \) and \( g_2 \) in \( G \), mapping to a basis of \( C \), such that \( g_1^p = 1 \).
(7) Set \( A' = \langle g_2 \rangle Z \) and \( C' = \langle g_1 \rangle \). Show that \( G = A' \times C' \) and that \( A' \) is either \( C_p^2 \) or \( C_{p^2} \).
(8) Classify the actions of \( C' \) on \( A' \) and thus show that \( G \) is one of the groups listed above.