Problem 1 Let $G = \text{Spec } k[u, u^{-1}]$ and let $\mu$ be the map $G \times G \to G$ corresponding to the map of rings $k[u, u^{-1}] \to k[u_1, u_1^{-1}, u_2, u_2^{-1}]$ which sends $u$ to $u_1u_2$. Let $\iota : \text{Spec } k \to G$ correspond to the map of $k$-algebras $k[u, u^{-1}] \to k$ sending $u \mapsto 1$. By action of $G$ on a scheme $X$, we mean a map of schemes $\alpha : G \times X \to X$ so that the two obvious maps $G \times G \times X \to X$ are equal and such that the composition $X \cong \text{Spec } k \times_k X \xrightarrow{\iota \times \text{Id}} G \times X \xrightarrow{\alpha} X$ is the identity.

Let $S$ be a $k$-algebra. In this problem, we will see that $\mathbb{Z}$-gradings on $S$ correspond to actions of $G$ on Spec $S$.

(a) Let $S = \bigoplus_{j=-\infty}^{\infty} S_j$ be a grading. Define a map $\alpha^* S \to S[u, u^{-1}]$ by $\alpha^*(f) = u^j f$ for $f \in S_j$. Show that $\alpha^*$ is a map of rings and the induced map $G \times \text{Spec } S \to \text{Spec } S$ is an action.

(b) Show that, if $f_j \in S_j$ and $\sum f_j = 0$ then each $f_j$ is $0$.

(c) Show that, if $f_i \in S_i$ and $f_{j} \in S_j$ then $f_i f_j \in S_{i+j}$.

Now, assume that $\alpha$ is an action.

(d) For any $f \in S$, let $\alpha^* f = \sum u^j f_j$. (Note that this is a finite sum.) Show that $f_j \in S_j$ and $f = \sum f_j$.

(e) Explain why we are done, i.e., why we have shown that an action of $G$ on Spec $S$ gives a $\mathbb{Z}$-grading of $S$.

Problem 2 Let $S$ be a $\mathbb{Z}$-graded ring. Let $\text{Homog}(S)$ be the set of homogenous primes of $S$. For a positive integer $d$, define $S^{(d)}$ to be the subring $\bigoplus_{j \geq 0} S_{jd}$ of $S$. In this problem, we will check some basic compatibilities between these rings. I've broken this into a lot of parts; they are all meant to be short.

(a) Let $I$ be a homogenous ideal of $S$. Show that $I$ is prime if and only if, for any homogenous elements $f$ and $g$, if $fg \in I$, then either $f \in I$ or $g \in I$.

(b) Let $I$ be a homogenous ideal of $S$. Show that $\sqrt{I} := \{f \in S : f^n \in I \text{ for some } n > 0\}$ is a homogenous ideal.

(c) Let $p$ be a homogenous prime ideal of $S$. Show that $p \cap S^{(d)}$ is a homogenous prime ideal of $S^{(d)}$.

(d) Let $q$ be a homogenous prime ideal of $S^{(d)}$. Show that $\sqrt{q} S$ is a homogenous prime ideal of $S$.

(e) Show that (c) and (d) provide inverse bijections between $\text{Homog}(S)$ and $\text{Homog}(S^{(d)})$.

(f) Let $f \in S_d$. Provide bijections between the following sets: $\{p \in \text{Homog}(S) : f \not\in p\}$, $\{q \in \text{Homog}(S^{(d)}) : f \not\in q\}$, $\text{Homog}(f^{-1}S^{(d)})$, $\text{Spec}((f^{-1}S^{(d)})_0)$, $\text{Spec}((f^{-1}S)_0)$.

The final composite bijection between $\{p \in \text{Homog}(S) : f \not\in p\}$ and $\text{Spec}((f^{-1}S)_0)$ is justified by Hartshorne under the statement “The properties of localization show that $\phi$ is bijective . . .” in the proof of Proposition 2.5; I wrote this exercise to understand that sentence. If you see a faster route, let me know!