Hints for this problem set are on the rear of the page. I think this is pretty hard, but doable, without them.

**Problem 1** In this problem, we’ll investigate the relationship between partitions of unity and sheaf cohomology vanishing. Let \((X, \mathcal{O})\) be a locally ringed space.

We say that \((X, \mathcal{O})\) and let \(U_i\) be a locally finite cover of \(U_i\). We define a **strong partition of unity** subordinate to \(U_i\) to be functions \(\phi_i \in \Gamma(\mathcal{O})\) such that \(\sum \phi_i = 1\) and, if \(z \in X \setminus U_i\), then \(\phi_i\) vanishes in the stalk \(\mathcal{O}_z\). We will say that \(\phi_i\) is a **weak partition of unity** subordinate to \(U_i\) if the same holds but we only require \(\phi_i\) to lie in the maximal ideal \(m_z \subset \mathcal{O}_z\). (These are not standard terms.) Clearly, strong partitions of unity implies weak partitions of unity.

(a) Suppose that \(X\) is paracompact (every open cover has a locally finite refinement) and regular (for any point \(z\) and any closed set \(K\), there is are open sets \(V \ni z\) and \(W \ni K\) with \(V \cap W = \emptyset\)). Suppose that every locally finite cover of \(X\) has a weak partition of unity. Show that every open cover of \(X\) has a locally finite refinement with a strong partition of unity.

From now on, suppose that every open cover of \(X\) has a locally finite cover with respect to which there is a strong partition of unity.

(b) Suppose that \(\mathcal{B} \to \mathcal{C}\) is a surjection of sheaves of \(\mathcal{O}\)-modules. Show that \(\Gamma(\mathcal{B}) \to \Gamma(\mathcal{C})\) is a surjection.

(c) Let \(\mathcal{A}\) be an arbitrary sheaf of \(\mathcal{O}\)-modules. Show that \(H^1(\mathcal{A}) = 0\). You may assume that \(\mathcal{A}\) injects into an injective \(\mathcal{O}\)-module \(\mathcal{I}\).

(d) Let \(\mathcal{A}\) be an arbitrary sheaf of \(\mathcal{O}\)-modules. Show that \(H^q(\mathcal{A}) = 0\). You may assume that every sheaf of \(\mathcal{O}\)-modules injects into an injective sheaf of \(\mathcal{O}\)-modules.

**Problem 2** Given a map of complexes \(A^\bullet \xrightarrow{f} F = B^\bullet\), define the cone complex \(\text{Cone}(A^\bullet \xrightarrow{f} B^\bullet)\) where \(\text{Cone}(A^\bullet \to B^\bullet)^q = A^{q+1} \oplus B^q\) and where the map \(A^{q+1} \oplus B^q \to A^{q+2} \oplus B^{q+1}\) is given by \((\begin{smallmatrix} \partial & (-1)^q \xi \end{smallmatrix})\). We have obvious maps of complexes \(B^\bullet \xrightarrow{j} \text{Cone}(A^\bullet \xrightarrow{f} B^\bullet) \xrightarrow{j'} A^{\bullet+1}\).

(a) Abbreviating \(\text{Cone}(A^\bullet \xrightarrow{f} B^\bullet)\) to \(K^\bullet\), show that we have a long exact sequence

\[
\cdots \to H^0(B^\bullet) \xrightarrow{j^*} H^0(K^\bullet) \xrightarrow{j^*} H^1(A^\bullet) \xrightarrow{j^*} H^1(K^\bullet) \xrightarrow{j^*} H^2(A^\bullet) \to \cdots,
\]

where we have labeled each arrow by the map of complexes which induces it. Note that \(H^q(A^{\bullet+1}) = H^{q+1}(A^\bullet)\), so \(j^*\) maps \(H^q(K^\bullet) \to H^{q+1}(A^\bullet)\).

Now suppose that we have three complexes \(A^\bullet \xrightarrow{\alpha} B^\bullet\) and \(B^\bullet \xrightarrow{\beta} C^\bullet\) and maps of complexes \(A^\bullet \xrightarrow{\alpha} f \xrightarrow{f} B^\bullet\) and \(B^\bullet \xrightarrow{\beta} C^\bullet\) such that \(0 \to A^q \xrightarrow{\alpha} B^q \xrightarrow{\beta} C^q \to 0\) is exact for all \(q\).

(b) Construct a map of complexes \(\text{Cone}(A^\bullet \xrightarrow{\alpha} B^\bullet) \to C^\bullet\) which induces isomorphisms \(H^q(\text{Cone}(A^\bullet \xrightarrow{\alpha} B^\bullet)) \cong H^q(C^\bullet)\) for all \(q\).

Combining the two problems, we have constructed a long exact sequence

\[
\cdots \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) \to H^1(A^\bullet) \to H^1(B^\bullet) \to H^1(C^\bullet) \to \cdots.
\]
Hint for 1(a): Let our cover be $X = \bigcup U_i$ and let $K_i = X \setminus U_i$. For every $z \in X$, choose a $U_i \ni z$ and choose disjoint opens $V_z \ni z$ and $W_z \supset K_i$. Consider show that weak partitions of unity subordinate to a locally finite refinement of $V_z$ are strong partitions of unity with respect to $U_i$.

Hint for 1(b): Lift $c \in \Gamma(C)$ to $b_i \in \mathcal{B}(U_i)$. After refining appropriately, take a strong partition of unity $\phi_i$ subordinate to $U_i$. Show that $\phi_i b_i$ extends to an element of $\Gamma(B)$, and that $\sum \phi_i b_i \mapsto c$.

Hint for 1(c): Use part (b) and play with the injection $\mathcal{A} \to \mathcal{I}$.

Hint for 1(d): Play with the injection $\mathcal{A} \to \mathcal{I}$.