

NOTES FOR JANUARY 6

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1. COURSE ORGANIZATION

The course website, <http://www.math.lsa.umich.edu/speyer/632/932.html>, has information about contacting Prof. Speyer, office hours, etcetera.

There will be two tasks for those enrolled in the course: There will be weekly problem sets, distributed on Tuesdays in class and due on the following Tuesday. You may collaborate on solving problems, but must write them up alone, in your own words.

The second responsibility is to take turns providing \TeX 'ed notes for the class, as I have done here. There is a template on the course webpage to get you started, and I can help you if you haven't used \TeX before. There will be an e-mail sent soon to arrange scribes for the first month.

There are a lot of people attending this class who are not enrolled. If you are one of them, e-mail me and I'll add you to the course e-mail list.

The textbook is *Hodge Theory and Algebraic Geometry I*, by Claire Voisin.

2. ABOUT THE COURSE

I grew up as an algebraic algebraic geometer, reading books by authors like Fulton, Eisenbud and Harris, and Hartshorne. In the last several years, I have been trying to learn the complex analytic perspective on algebraic geometry: How to study complex varieties using smooth functions and their derivatives, and how to use this sort of data to extract topological data. This course is my attempt to present that perspective.

I expect people to have background in three fields, but the amount I need from each is small.

The first is complex analysis. You should know:

- What an analytic (also known as holomorphic) function is.
- The relation between analytic functions and convergent power series.
- How to compute integrals of analytic functions using residues.

The second is smooth manifolds. You should know:

- What a manifold is.
- What a smooth manifold is.
- What a tangent vector, and a tangent vector field are.
- What a differential form is. I'll review this on Tuesday, but I'll probably go way too fast for someone who hasn't seen this before.

The third is algebraic topology, by which I basically mean cohomology. (There may be some π_1 in this course, there will not be any higher homotopy groups.) I asked a senior faculty member what students learn about cohomology in the standard course here. I was told "They've learned what cohomology is, but they've forgotten it."

So, let me remind you of a few notions. What I want is that, as I bring these subjects up, you remember them.

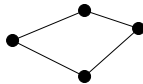
Let $A^0 \rightarrow A^1 \rightarrow A^2 \rightarrow \dots$ be a sequence of abelian groups, and maps between them. It is called a **complex** if, for every k , the composition $A^k \rightarrow A^{k+1} \rightarrow A^{k+2}$ is 0. So, for any complex, the image of $A^{k-1} \rightarrow A^k$ is contained in the kernel of $A^k \rightarrow A^{k+1}$. The complex is called **exact** if, for every k , the image is equal to the kernel. The failure of exactness is measure by cohomology: The k -th cohomology group $H^k(A^\bullet)$ is the quotient $\text{Ker}(A^k \rightarrow A^{k+1})/\text{Im}(A^{k-1} \rightarrow A^k)$.

If X is a topological space, there is a complex $C^\bullet(X)$ called the cochain complex of X . Its cohomology $H^\bullet(X)$ is a topological (in fact, homotopy) invariant of X . I want you to have a basic memory of this construction, and have seen its basic properties.

3. COHOMOLOGY FROM MANY PERSPECTIVES

As a preview of this year's subject matter, I will discuss the cohomology of the circle from many perspectives. The circle, S^1 , is of course the topological space obtained by taking an interval $[0, 1]$ and identifying the endpoints; we can also construct it as the set of $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 = 1$.

3.1. Triangulate S^1 . For example, we can make it into a quadrilateral:



We then have a cochain complex, where the first space is functions on vertices and the second is functions on edges. Given a function ϕ on the vertices, and given an edge e between vertices u and v , we have $(d\phi)(e) = \phi(u) - \phi(v)$. Explicitly, the cochain complex is

$$\mathbb{Z}^4 \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \mathbb{Z}^4.$$

This is a complex: All but one map is zero, so the condition that composing any two maps is 0 is trivially obeyed. We want to compute the kernel and the cokernel of this map.

Clearly, a function on vertices is in the kernel if and only if it is constant. So

$$H^0(S^1) = \{\text{constant functions}\} \cong \mathbb{Z}.$$

To compute H^1 , note the characteristic function of one edge is equivalent to the characteristic function of any other in the cokernel. So $H^1(S^1)$ is spanned by the class of any edge and

$$H^1(S^1) \cong \mathbb{Z}.$$

Comment not made in class: Given a function ψ on edges, its class in $H^1(S^1)$ is $\sum_{e \in \text{Edges}} \psi(e)$. This sum will become an integral in other theories of cohomology.

3.2. In singular cohomology, we form an infinite cochain complex. The zeroth term is all functions from S^1 to \mathbb{Z} : no measurability, no continuity, just all functions whatsoever! The first term is even worse: All \mathbb{Z} -valued functions on the set of all continuous maps from $[0, 1]$ to S^1 . I used to think topology was really hard, because topologists spent all their time thinking about sets of such large size. We won't see much of this construction.

Correction to class: I accidentally implied that the singular cochain complex stops after the C^1 term. That, of course, is not true. You also have to consider \mathbb{Z} -valued functions from the set of all continuous maps from the triangle to S^1 , and \mathbb{Z} -valued functions from the set of all continuous maps from the tetrahedron to S^1 , and so forth.

3.3. In the de Rham approach to cohomology, we again consider a complex with 2 terms. The first one is smooth functions from S^1 to \mathbb{R} , which we denote $C^\infty(S^1)$. The second space, properly speaking, is $\Omega^1(S^1)$, the space of smooth 1-forms on S^1 . Since I won't be reviewing 1-forms until tomorrow, those of you who aren't comfortable with them can just use the isomorphism $\Omega^1(S^1) \cong C^\infty(S^1)$. Our map sends the function f to the 1-form df or, more explicitly, sends $f(\theta)$ to $\partial f / \partial \theta d\theta$. If you don't know what $d\theta$ means, ignore it.

Again, we want to compute the kernel and cokernel of this map. A function whose derivative is 0 is a constant. So

$$H_{DR}^0(S^1) = \{\text{constant functions } S^1 \rightarrow \mathbb{R}\} \cong \mathbb{R}.$$

What about the cokernel? Given $g : S^1 \rightarrow \mathbb{R}$, we want to know whether there exists a function f with $f' = g$. The way to find such a function is to integrate: We try setting

$$g(\theta) = \int_0^\theta f(\tau) d\tau.$$

But this might have a jump discontinuity at 0. We only get a smooth f if $\int_0^{2\pi} f(\tau) d\tau = 0$.

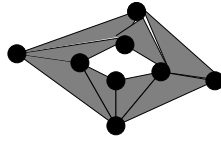
So the image of d is functions with integral 0, and

$$H_{DR}^1(S^1) = \{\text{functions } S^1 \rightarrow \mathbb{R}\} / \{\text{such functions with } \int f = 0\} \cong \mathbb{R}.$$

Given a function g , its class in H^1 is $\int g$.

3.4. For our next examples, we switch targets from S^1 to $\mathbb{R}^2 \setminus \{(0,0)\}$. These two spaces are deformation retracts of each other, so they should have the same cohomology. I'll call this space T .

Triangulating a noncompact space is a slightly subtle notion, so I'll cheat and triangulate the closed annulus instead.



We now have three nontrivial terms in our cochain complex – functions on vertices, functions on edges and functions on triangles.

$$\mathbb{Z}^8 \rightarrow \mathbb{Z}^{16} \rightarrow \mathbb{Z}^8.$$

I won't write out the boundary maps but, if you do so, you should get that $H^0(T) \cong \mathbb{Z}$, $H^1(T) \cong \mathbb{Z}$ and $H^2(T) \cong 0$.

3.5. Let's redo the de Rham computation, working on the punctured plane this time. We now have three spaces: $C^\infty(T)$, $\Omega^1(T)$ and $\Omega^2(T)$. These are isomorphic to $C^\infty(T)^{\oplus 2}$ and $C^\infty(T)$.

Our maps are

$$\begin{array}{ccccc} C^\infty(T) & \xrightarrow{d} & C^\infty(T)^2 & \xrightarrow{d} & C^\infty(T) \\ f(x,y) & \mapsto & (\partial f/\partial x, \partial f/\partial y) & & \\ & & (f(x,y), g(x,y)) & \mapsto & \partial f/\partial y - \partial g/\partial x \end{array}$$

Those of you who with a background in physics might call these maps “grad” and “curl”. Observe that $f(x,y) \mapsto (\partial f/\partial x, \partial f/\partial y) \mapsto \partial^2 f/(\partial y \partial x) - \partial^2 f/(\partial y \partial x) = 0$, so this is a complex.

So, H^0 is the functions with gradient 0. This is the constant functions. (More precisely, the locally constant functions, but T is connected.)

I leave H^2 as an exercise. The answer is that every function on T is a curl; so $H^2 = 0$.

The interesting question is H^1 . Given a vector field v (more precisely, a 1-form, but I'll use the lower level terminology for now) with $\nabla \times v = 0$, we want to know whether $v = \nabla g$.

We can try to build g by integrating v . Say

$$g(x,y) = \int_{(1,0)}^{(x,y)} v$$

along some path γ . The trouble is that this integral might depend on γ .

Now, because $\nabla \times v = 0$, moving γ by a homotopy can't change the integral. But two paths which go around the origin in different ways might have genuinely different integrals. In other words, the obstruction to v being a gradient is if $\oint v \neq 0$, where \oint is taken around the origin. So

$$H_{DR}^1(T) = \{\text{vector fields } v \text{ with } \nabla \times v = 0\} / \{\text{vector fields } v \text{ with } \nabla \times v = 0 \text{ and } \oint v = 0\}.$$

This quotient is \mathbb{R} , with the map sending a vector field v to $\oint v$.

3.6. The space T is isomorphic to $\mathbb{C} \setminus \{0\}$. I'll call it \mathbb{C}^* when I'm thinking of it this way.

Let's try setting up de Rham cohomology again, but using only analytic functions. So the terms in our cochain complex are $\{\text{analytic functions on } \mathbb{C}^*\}$ and $\{\text{holomorphic } (1,0) \text{ forms on } \mathbb{C}^*\}$. Since I don't expect you to know what $(1,0)$ forms are, it is fortunate that the second term is isomorphic to $\{\text{analytic functions on } \mathbb{C}^*\}$. Our differential map sends an analytic function $f(z)$ to the analytic function $(\partial f/\partial z)$.

Once again, a function whose derivative is zero is a constant. So H^0 is constant functions and we get $H^0(\mathbb{C}^*) = \mathbb{C}$.

For H^1 , we have $g(z)$, an analytic function on \mathbb{C}^* , and we want to know whether there is some f such that $\partial f/\partial z = g$. Just like in the case of de Rham cohomology for the punctured plane, we would like to take $f(z) = \int_1^z g(w)dw$, but we have to worry about what path to take the integral along. Since g is analytic, only the homotopy class of the path matters.

So, once again, we need to worry about whether $\oint g(w)dw$ vanishes. We see that the image of d is analytic functions on \mathbb{C}^* with residue 0. So $H^1(\mathbb{C}^*) = \mathbb{C}$.

Correction of what I said in class: For some reason, I referred to this as the Dolbeault cohomology of \mathbb{C}^* . The Dolbeault cohomology is what you get when you map *smooth* \mathbb{C} -valued functions to *smooth* \mathbb{C} -valued $(0,1)$ forms by the $\bar{\partial}$ map, which we will define soon. In this case,

$H_{\text{Dolbeault}}^1(\mathbb{C}^*) = 0$, as we will prove in a few weeks. This fact is important, and is an important lemma when relating the computations I am doing here to topological cohomology, but it is very untrue that Dolbeault cohomology equals topological cohomology. All my mathematical statements were right, but for some reason the wrong term came to mind.

The general result is that, if X is a smooth *affine* complex variety of *complex* dimension d , then $H^k(X)$ vanishes for $k > d$, and can be computed by writing down the de Rham complex using analytic forms. We will be proving this (modulo some things I may have to gloss over).

In fact, X is homotopic to a CW complex of dimension d .

A nice example, for those of you who have seen elliptic curves before, is an elliptic curve with one point removed. This is homotopic to a wedge of two circles, so the homology groups should be $H^0 = \mathbb{C}$, $H^1 = \mathbb{C}^2$ and $H^2 = 0$. If you know how to work with elliptic functions, you should be able to check that this is what you get if you write down the de Rham complex for analytic functions, as I did above.

For projective varieties, it is not true that cohomology vanishes above degree d . But it is true that there is a way to extract the cohomology from purely topological data. We will discuss this, in fact, it is our main topic.

3.7. Finally, let me present one more variant. Instead of working with all analytic functions on \mathbb{C}^* , let's just work with polynomials. So the zero-eth term in our cochain complex will be $\mathbb{C}[z, z^{-1}]$, and we map $\mathbb{C}[z, z^{-1}]$ to itself by taking derivatives.

Things work out exactly the same. A polynomial has derivative 0 if and only if it is a constant. A Laurent polynomial is a derivative of a Laurent polynomial if and only the coefficient of z^{-1} is zero. So, once again, $H^0(\mathbb{C}^*)$ and $H^1(\mathbb{C}^*)$ are each \mathbb{C} .

This is an example of Serre's GAGA (*Géométrie Algébrique et Géométrie Analytique*) principle: Algebraic and complex analytic computations give the same answers, when you ask the right questions. I hope to return to this before the end of the course.

This formulation of cohomology, and the original one in terms of triangulations, are by far the best suited to automatic computation. And there are many spaces where it is reasonable to compute the algebraic de Rham cohomology in this fashion, but it is insane to attempt to triangulate them. For example, take \mathbb{C}^3 and remove the union of 6 complex hyperplanes.

4. OVERVIEW OF THE COURSE

This lecture was an overview of the cohomology theories we will be talking about:

- Topological (also known as Betti) cohomology, usually described in terms of triangulations
- De Rham cohomology, using smooth functions
- Analytic de Rham cohomology¹, using analytic functions
- Algebraic de Rham cohomology, using polynomials

We will spend the next few weeks talking about the relation of the first two. We will spend most of the course talking about the relation of the second and third. I hope to say a bit about the last.

5. ONE FINAL, UNPLANNED, REMARK

Suppose you were a number theorist, so you thought about $\text{Spec } \mathbb{Q}[z, z^{-1}]$ instead of \mathbb{C}^* . Then it would be natural to compute algebraic de Rham cohomology with rational coefficients. You would get that $H^1 = \mathbb{Q}$, with basis dz/z .

On the other hand, if you were a topologist, you could also consider topology with \mathbb{Q} coefficients, working from a triangulation. Again, $H^1 = \mathbb{Q}$, with obvious basis the class of a single edge in the triangulation.

The ratio of these two basis elements, if you get them to live in the same space, is $\oint dz/z = 2\pi i$. This is the first example of what is called a *period*, a number which comes up when you try to relate topological cohomology to algebraic de Rham with rational coefficients. The fact that periods are often transcendental indicates that this isomorphism must be very mysterious.

¹I'm not sure if this is the standard name