PROBLEM SET 11
DUE APRIL 12, 2011

1. (Riemann-Roch for curves) Let $X$ be a compact complex curve and let $p$ be a point of $X$. Let $\mathbb{C}_p$ be the skyscraper sheaf at $p$, meaning that $\mathbb{C}_p(U) = \mathbb{C}$ if $p \in U$ and 0 if $p \notin U$. Let $D$ be any divisor on $X$.

   (1) Show that there is a short exact sequence $0 \to \mathcal{O}(D) \to \mathcal{O}(D + p) \to \mathbb{C}_p \to 0$.
   (2) Define $\chi(D) = \dim H^0(X, \mathcal{O}(D)) - \dim H^1(X, \mathcal{O}(D))$. Show that $\chi(D + p) = \chi(D) + 1$.
   Deduc that there is a constant $k$ such that $\chi(D) = \deg(D) + k$.
   (3) Considering $D = \emptyset$, compute $k$.
   (4) Suppose there is a divisor $K$ such that $\mathcal{H}^1 \cong \mathcal{O}(K)$. Compute $\deg(K)$.

2. Let $X$ be a compact complex curve. The point of this problem is to understand the map $\text{NS} : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ explicitly. Let $D$ be a divisor in $X$, in other words, $D = \sum a_i D_i$ for some finite set of points $D_i$ in $X$. Choose a triangulation of $X$ where all the $D_i$ are at vertices and where every triangle contains at most one $D_i$. For each vertex $v$ of the triangulation, let $U_v$ be the open neighborhood of $v$ described in the notes for January 20. We’ll compute Čech cohomology for that open cover.

   (1) Describe $\mathcal{O}(D)$ as a Čech cocycle $U_{ij} \cap U_{jk} \mapsto f_{ij} \in \mathcal{O}^*(U_{ij} \cap U_{jk})$.
   (2) Describe $(\text{NS})(\mathcal{O}(D))$ as an actual Čech cocycle for $H^2(X, \mathbb{Z})$.
   (3) Recall that we can integrate to get an integer $\int_X (\text{NS})(\mathcal{O}(D))$ (see Problem 5.(2), Set 3.) What integer do we get?

3. Let $X$ be a complex compact curve. Let $U$ be an open subset of $X$ which is isomorphic to a contractible subset of $\mathbb{C}$. Let $p$ and $q$ be points in $U$, and $\gamma$ a path from $p$ to $q$ in $U$.

   We know that $\mathcal{O}(p - q)$ is in $\text{Pic}^0(X)$, and we know that we have a surjection from $H^{0,1}(X)$ to $\text{Pic}^0(X)$. In the first part of this problem, we will find a $(0,1)$-form $\eta$ which maps to $\mathcal{O}(p - q)$.

   Choose a connected open set $V$, within $U$, and containing $\gamma$. Let $\lambda$ be a smooth function on $U$ such that, on $U \setminus V$, we have $\lambda = \log((z - p)/(z - q))$ with the branch cut chosen along $\gamma$. Let $U'$ be an open set with $U' \cup U'' = C$ and $V \cap U'' = \emptyset$.

   (1) Show that there is a $\overline{\partial}$-closed $(0,1)$-form $\eta$ which is $\overline{\partial}\lambda$ on $U$ and is 0 on $U'$.
   (2) Show that the Čech cocycle $U \cap U' \mapsto \log((z - p)/(z - q))$ represents the same class in $H^1(\mathcal{O})$ as does $\eta$.
   (3) By the preceding, $U \cap U' \mapsto e^{\log((z - p)/(z - q))}$ is the class in $H^1(\mathcal{O}^*)$ to which $\eta$ maps. Show that this co-cycle also represents $\mathcal{O}(p - q)$.

   Now, we know that $H^{0,1}(X)$ is dual to $H^{1,0}(X)$, by Serre duality. Explicitly, the class $\eta$ gives the map $\omega \mapsto \int_X (\eta \wedge \omega)$ from $H^{1,0}(X) \to \mathbb{C}$. In the second part of this problem, we study $\int_X (\eta \wedge \omega)$.

   (4) Show that $\int_X (\eta \wedge \omega) = \int_U (\eta \wedge \omega) = \int_{\partial U} \lambda \omega$.
   (5) Choose a metric on $X$. Let $N(\gamma, \epsilon)$ be the points of $X$ which are within $\epsilon$ of some point of $\gamma$. Let $\epsilon$ be small enough that $N(\gamma, \epsilon) \subset U$. Show that $\int_{\partial N(\gamma, \epsilon)} \lambda \omega = \int_{\partial U} \lambda \omega$.
   (6) Show that
   \[
   \lim_{\epsilon \to 0} \int_{\partial N(\gamma, \epsilon)} \lambda \omega = (2\pi i) \int_{\gamma} \omega.
   \]
Remark: More generally, let $p_1, \ldots, p_N$ and $q_1, \ldots, q_N$ be points of $X$, and let $\gamma_i$ be a path in $X$ from $p_i$ to $q_i$. Then the map $\omega \mapsto (2\pi i) \sum \int_{\gamma_i} \omega$ is an element of $H^{1,0}(X)^*$ which represents $\mathcal{O}(\sum p_i - \sum q_i)$. As a corollary: we obtain the Abelian-Jacobi theorem: The line bundle $\mathcal{O}(\sum p_i - \sum q_i)$ is trivial if and only if the linear function $\omega \mapsto (2\pi i) \sum \int_{\gamma_j} \omega$ is in the lattice of maps of the form $\omega \mapsto (2\pi i) \int_{\sigma} \omega$, for some $\sigma \in H_1(X, \mathbb{Z})$.

4. Let a closed submanifold of a polydisc, with $H^2(X, \mathbb{Z}) = 0$.

   (1) Show that $H^1(X, \mathcal{O}^*)$ is trivial.

   (2) Let $D$ be a smooth hypersurface in $X$. Show that there is a holomorphic function on $X$ whose zeroes are precisely $D$.

5. What is wrong with the following argument? Let $E$ be the hypersurface $y^2 = x^3 - x$ in $\mathbb{C}^2$. As we’ve computed before, $E$ only has cohomology in degree $\leq 1$, so $H^2(E, \mathbb{Z})$ is trivial, and $H^1(E, \mathcal{O}) = 0$ as $E$ is a submanifold of $\mathbb{C}^2$. So, by the previous problem, for any point $z$ in $E$, there is a holomorphic function $f$ on $E$ that vanishes once at $z$ and nowhere else.

   Let $X$ be the compactification of $E$ constructed in Problem Set 9, Problem 5, and let $\infty$ be the extra point of $E \setminus X$. Since $f$ has a single zero on $E$, it must have a simple pole at $\infty$. So $\mathcal{O}(z - \infty)$ is trivial in Pic$(X)$. Since $z$ was arbitrary, this shows that every degree 0 divisor $D$ has $\mathcal{O}(D)$ trivial in Pic$(X)$, and thus Pic$^0(X)$ is trivial.