PROBLEM SET 6
DUE FEBRUARY 24, 2011 – NOTE UNUSUAL DATE

1. Let $L$ be a trivial complex line bundle on $X$, some real manifold. Let $\nabla$ be a connection on $L$. If we choose an isomorphism between $L$ and the product line bundle $\mathbb{C} \times X$, then sections of $L$ can be identified with functions $X \to \mathbb{C}$. We’ll write $\alpha(s)$ for the function corresponding to $s$, we will also write $\alpha$ for the identification of sections of $L \otimes \Omega^1$ with 1-forms.

We showed in class that there is a one form $\omega$ such that

$$\alpha(\nabla(s)) = d\alpha(s) + \alpha(s) \omega.$$  

(1) In terms of $\omega$, what is the map $\nabla^2 : C^\infty \otimes L \to \Omega^2 \otimes L$? When is $\nabla$ integrable?

(2) Suppose we choose a different trivialization $\beta$ of $L$, such that $\beta(s) = g \alpha(s)$, where $g$ is some function $X \to \mathbb{C}^\times$. In the new coordinates, let $\beta(\nabla(s)) = d\beta(s) + \beta(s) \eta$. What is the relation between $\omega, \eta$ and $g$?

2. Let $M$ be a connected smooth manifold and $V$ a smooth $\mathbb{R}$ vector bundle over $M$. Suppose that, for each fiber $V_x$, we have an inner product $(\cdot, \cdot)$ on $V_x$. Let $\nabla$ be a connection on $V$. Suppose that, for any two sections $\sigma, \tau$ of $V$, and any vector field $X$, we have the equality

$$X(\sigma, \tau) = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle.$$  

Let $\sigma$ be a section of $V$ which is $\nabla$-constant, meaning that $\nabla(\sigma) = 0$. Show that $\langle \sigma, \sigma \rangle$ is constant.

3. Let $M$ be a connected smooth manifold and $V$ a smooth $\mathbb{R}$ vector bundle over $M$. Suppose that, for each fiber $V_x$, we have a linear endomorphism $E : V_x \to V_x$. Let $\nabla$ be a connection on $V$. Suppose that, for any section $\sigma$ of $V$, and any vector field $X$, we have the equality

$$\nabla_X (E \sigma) = E \nabla_X (\sigma).$$  

Let $\sigma$ be a section of $V$ which is $\nabla$-constant, meaning that $\nabla(\sigma) = 0$. Show that $E \sigma$ is also $\nabla$-constant.

4. This is a continuation of problems 3 and 4 from the previous problem set. Recall that $p$ is a polynomial of degree $2g + 1$ without repeated roots, and $W$ is the hypersurface $y^2 = p(x)$ in $\mathbb{C}^2$. In that problem, we found a holomorphic $(1, 0)$-form $\omega$ on $W$, given by $\omega = dx/(2y) = dy/p'(x)$. The holomorphic $(1, 0)$-forms on $W$ are of the form $f \omega$ for some holomorphic $f$.

(1) Let $g(x)$ be a holomorphic function on $\mathbb{C}$. Express $dg$ as multiple of $\omega$.

(2) For any entire function $u(x)$, show that $u(x) y \omega$ is of the form $dg$ for some $g(x)$.

(3) Let $h(x)$ be a holomorphic function on $\mathbb{C}$. Express $d(hy)$ as a multiple of $\omega$.

(4) Let $B$ be the vector space of polynomials $v(x)$ such that there is a polynomial $h(x)$ with $d(h(x)y) = v(x) \omega$. Show that $\mathbb{C}[x]/B \cong \mathbb{C}^{2g}$.

(5) **Fairly hard bonus question:** Same as the above question, with $v$ and $h$ entire. When I attempted this, it took some fairly messy analysis; I’m curious whether you can find a clean argument.

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1. Most mathematicians would write $d(\sigma, \tau) = \langle \nabla \sigma, \tau \rangle + \langle \sigma, \nabla \tau \rangle$. Exercise for those who want to work it out: Explain and justify the abuses of notation in this equation.

2. As in the last footnote, the normal way to write this would be $\nabla(E \sigma) = E \nabla(\sigma)$. Again, what abuses of notation is this concealing?