PROBLEM SET 8
DUE MARCH 22, 2011

WORKING WITH HERMITIAN FORMS IN COORDINATES

1. On \( \mathbb{C}^2 \), let the coordinates be \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \). Consider the Hermitian form \( pdz_1 \otimes \overline{dz_1} + (q + ir)dz_1 \otimes \overline{dz_2} + (q - ir)dz_2 \otimes \overline{dz_2} + sdz_2 \otimes \overline{dz_2} \) for real numbers \( p, q, r \) and \( s \).

Expand this form as \( g - i\omega \), and express \( g \) and \( \omega \) in the \( x_1, x_2, y_1, y_2 \) coordinates. Check directly that \( g \) is symmetric and \( \omega \) is antisymmetric.

SOME PRACTICE USING “NICE” COORDINATES

2. Let \( g - i\omega \) be a Kähler form and let \( * \) be the Hodge star with respect to \( g \). Show that \( d^* \omega = 0 \).

3. Let \( X \) be a compact Kähler manifold. Let \( L : \Omega^k \rightarrow \Omega^{k+2} \) be the map \( \eta \rightarrow \omega \wedge \eta \). Let \( \Lambda = *^{-1}L* \). Show that, for \( \eta \in \Omega^k \), we have \( \Lambda(L\eta) - L(\Lambda\eta) = (n-k)\eta \).

AN ALGEBRAIC CONSEQUENCE OF HODGE’S THEOREM

4. Let \( X \) be a compact Kähler manifold. Let \( \mathcal{H}^p \) and \( \mathcal{Z}^{p+1} \) be the sheaves of holomorphic \( p \)-forms and \( \partial \)-closed holomorphic \( (p+1) \)-forms respectively.

   (1) For every \( q \), show that the map \( H^q(X, \mathcal{H}^p) \xrightarrow{\partial} H^q(X, \mathcal{Z}^{p+1}) \) is zero. (Hint: This is really simple using harmonic representatives.)

   (2) For \( X \) a compact complex manifold, even without the Kähler condition, show that \( H^0(X, \mathcal{H}^0) \rightarrow H^0(X, \mathcal{H}^1) \) is zero.

5. The point of this exercise is to explore the consequences of problem 4. Just using the fact that \( H^q(X, \mathcal{H}^p) \xrightarrow{\partial} H^q(X, \mathcal{Z}^{p+1}) \) is zero, show the following:

   (1) There is a short exact sequence \( 0 \rightarrow H^0(X, \mathcal{H}^1) \rightarrow H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{H}^0) \rightarrow 0 \).

   (2) There is a filtration \( 0 \subseteq F_1 \subseteq F_2 \subseteq H^2(X, \mathbb{C}) \) such that \( F_1 \cong H^0(X, \mathcal{H}^2) \), \( F_2/F_1 \cong H^1(X, \mathcal{H}^1) \) and \( H^2(X, \mathbb{C})/F_2 \cong H^2(X, \mathcal{H}^0) \).

THE FORMAL CONSEQUENCES OF THE KÄHLER IDENTITIES

6. The point of this exercise is to study a formal algebraic model; this will correspond to the action of the various differential operators on the \( \lambda \)-eigenspaces of \( \Delta \) for \( \lambda > 0 \).

   Let \( \lambda > 0 \). Let \( V^{p,q} \), for \( 0 \leq p, q \leq n \) be \( (n+1)^2 \) finite dimensional vector spaces with maps \( \partial, \overline{\partial}, \partial^* \) and \( \overline{\partial}^* \) between them shifting degrees in the obvious manners. Suppose that

   (1) \( \partial^2, \overline{\partial}^2, (\partial^*)^2 \) and \( (\overline{\partial}^*)^2 \) are all zero.

   (2) \( \partial\overline{\partial} + \partial\overline{\partial}^* + \overline{\partial}^* \partial, \partial^* \overline{\partial} + \overline{\partial} \partial^* + \overline{\partial}^* \partial^* \) and \( \partial^* \overline{\partial} + \overline{\partial} \partial^* \) are all zero.

   (3) \( \partial\partial^* + \partial^* \partial = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial} = \lambda \mathrm{Id} \).

Let \( Z^{p,q} \) be the subspace of \( V^{p,q} \) where \( \partial^* \) and \( \overline{\partial}^* \) are zero.

   (1) Show that \( Z^{p,q} \) is isomorphic to \( \partial Z^{p,q} \) and to \( \overline{\partial} Z^{p,q} \).

   (2) Show that \( Z^{p,q} \) is isomorphic to \( \partial \overline{\partial} Z^{p,q} \). Note: This requires nontrivial diagram tracing.

   (3) Show that \( V^{p,q} \cong Z^{p,q} \oplus \partial Z^{p-1,q} \oplus \overline{\partial} Z^{p,q-1} \oplus \partial \overline{\partial} Z^{p-1,q-1} \).