NOTES FOR APRIL 5TH

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1. The map \( H^k(X, \mathbb{Z}) \to H^k_{DR}(X, \mathbb{C}) \)

Given a geometric cocycle, how does it decompose under Hodge decomposition?

The bridge between the two is deRham cohomology through the following sequence of maps:

\[
H^k(X, \mathbb{Z}) \to H^k(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^k(X, \mathbb{C}) \to H^k_{DR}(X, \mathbb{C}).
\]

The matrix form of this map is more complicated and we first study the following easier map on a smooth n-dimensional manifold.

\[
H^k(X, \mathbb{Z}) \to Hom(H^k_{DR}(X), \mathbb{C})
\]

\[
\sigma \mapsto \omega \mapsto \int_\sigma \omega
\]

On Problem set 3, we did some non-trivial chasing through diagrams and checked this for \( k = 1, 2 \). We know from algebraic topology that \( H^k(X, \mathbb{C}) \) is dual to \( H^k(X, \mathbb{C}) \). \( Hom(H^k_{DR}(X, \mathbb{C})) \) is dual to \( H^k_{DR}(X) \) by definition. Therefore

\[
H^k_{DR}(X) \cong H^k(X, \mathbb{C})
\]

is the matrix transpose of the above isomorphism, i.e. mapping \( \omega \) to \( \sigma \mapsto \int_\sigma \omega \). The messy one is to invert this matrix to get the isomorphism from \( H^k(X, \mathbb{C}) \) to \( H^k_{DR}(X) \). The clever way is as follows. Assume \( X \) is compact, oriented of dimension \( m \). First, we have Poincare duality, as proved in (for example) Hatcher’s book:

\[
H^k(X, \mathbb{Z}) \cong H_{m-k}(X, \mathbb{Z})
\]

by capping with the fundamental class \([X]\). And we have Poincare duality as proved in this class: \( H^m_{DR}(X) \to H^{m-k}_{DR}(X)^* \) by \( \alpha \mapsto (\beta \mapsto \int_X \alpha \wedge \beta) \). These fit together into the right hand of the following diagram:

\[
\begin{array}{cccccc}
H^k_{DR}(X) & \cong & H^{m-k}_{DR}(X)^* & \longrightarrow & H^{p,q}(X) \\
\downarrow & & \downarrow & & \downarrow \\
H^k(X, \mathbb{Z}) & \overset{(\cdot)^{-1}}\longrightarrow & H^k_{DR}(X) & \cong & \cong & \cong \\
\downarrow & & \downarrow & & \downarrow \\
H_{m-k}(X, \mathbb{Z}) & \longrightarrow & H^{m-k}_{DR}(X) & \longrightarrow & H^{n-p,n-q}(X)^* \\
\end{array}
\]

This gives us a recipe to go from \( H^k(X) \) to \( H^k_{DR}(X) \), going down, right, up. That is to say, if \( \alpha \in H^k(X, \mathbb{Z}) \) maps to \( \eta \in H^k_{DR}(X) \), then, for any \( \theta \in H^{m-k}_{DR}(X) \), we have

\[
\int_X \eta \wedge \theta = \int_{\alpha \cap [X]} \theta.
\]
Now, suppose we want to understand how this all ties into Hodge decomposition. Suppose that $X$ is compact Kähler, of complex dimension $n$. We then need the right hand side of the above diagram. The projection $H^k_{DR} \to H^{p,q}(X)$ is dual to the inclusion $H^{n-p,n-q}(X) \to H^{2n-k}$. In other words, if $\alpha$ maps to $\eta$ in $H^{p,q}(X)$ then, for any $(n-p, n-q)$-form $\eta$, we have the above equality.

2. Line Bundles

Line bundles are rank one holomorphic bundles. They are usually given either with glueing data or by a divisor.

Let’s start with glueing data: let $L$ be a holomorphic line bundle over $X$, a complex manifold. Then the glueing data of $L$ is data or by a divisor.

Now, suppose we want to understand how this all ties into Hodge decomposition. Suppose $X$ is compact Kähler, then $\Pic(X)$ can be an injection. In this way, we can have compact Kähler manifolds with trivial $\NS(X)$.
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Compatibility of various maps

Hunter Brooks raised a question. In the above discussion, we got maps from $H^2(X, \mathbb{Z})$ to $H^2(X, \mathcal{O})$ in 2 ways, one is through the exponential sequence, the other by tensoring the cohomology with $\mathbb{C}$, then projection by Hodge decomposition. Hunter’s question was whether these two maps are the same.

The inclusion of sheaf $\mathbb{Z}$ in $\mathcal{O}$ can be factored as $\mathbb{Z} \xrightarrow{2\pi i} \mathbb{C} \rightarrow \mathcal{O}$. Our first map is the map on $H^2$ derived from the composite of these maps. By functoriality of cohomology, we can break this up into $H^2(\mathbb{Z}) \rightarrow H^2(\mathbb{C}) \rightarrow H^2(\mathcal{O})$. The first map is precisely the “tensor with $\mathbb{C}$” part of our second map. So the thing that remains to see is that the projection $H^2(X, \mathbb{C}) \rightarrow H^2(X, \mathcal{O})$ coming from the Hodge decomposition is the one coming from the map of sheaves $\mathbb{C} \rightarrow \mathcal{O}$. This is indeed true; it is perhaps easiest to think about this in terms of harmonic forms.

Even if $X$ is not compact Kahler, we have:

$$
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
Z & Z \\
\downarrow & \downarrow \\
0 & 2\pi i \\
\downarrow & \downarrow \\
\mathbb{C} & 0 \\
\downarrow & \downarrow \\
0 & \mathbb{C}^* \\
\downarrow & \downarrow \\
0 & 0
\end{array}
$$

where $Z^1$ is closed $(1,0)$-forms. The rows and columns are exact and the diagram commutes.

There are two ways to go from $H^1(\mathcal{O}^*)$ to $H^2(X, \mathbb{C})$ that we can observe from the previous diagram. One way is $H^1(\mathcal{O}^*) \rightarrow H^1(Z^1) = H^1(\mathbb{C}) \rightarrow H^2(\mathbb{C})$, using the horizontal exact sequences. The other is $H^1(\mathcal{O}^*) \rightarrow H^2(\mathbb{Z}) = H^2(\mathbb{C}) \rightarrow H^2(\mathcal{O})$, using the vertical exact sequences. These maps are equal, which is a general homological fact.

NS($X$) is the image of $H^1(\mathcal{O}^*)$ in $H^2(\mathbb{Z})$; we thus see that we can describe NS($X$) as the subgroup of $H^2(X, \mathbb{Z})$ consisting of those classes that have image in $H^2(X)$ representable by a closed $(1,1)$-form.

Divisors and line bundles

If $X$ is compact Kähler then, as discussed at the beginning of lecture, $NS(X)$ is the subset of $H^2(X, \mathbb{Z}) = H_{2n-2}(X, \mathbb{Z})$ corresponding to codimension 2 cycles which integrate to 0 against all closed $(n, n-2)$-forms.

Similarly,

$$
\text{Pic}^0(X) \cong H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \cong H^{0,1}(X)/H^1(X, \mathbb{Z}) \cong H^{n,n-1}(X)^*/\{\text{functionals of the form } \int_\sigma \text{ for } \sigma \in H_{2n-1}(X, \mathbb{Z})\}.
$$
In particular, if $D \subset X$ is a complex submanifold of dimension $n - 1$, then $\int_D \omega = 0$ for any $\omega \in H^{n,n-2}(X)$ as $\omega|_D = 0$. So there should be some line bundle whose Neron-Severi class is $D$ (which will prove two classes later).

We can also give a geometric way of thinking of the description of $\text{Pic}^0(X)$ as $H^{n,n-1}(X)^*$ modulo integration over closed $2n - 1$ cycles. Specifically, suppose $D \subset X$ has class 0 in $H_{2n-2}(X)$, so $D = \partial M$, where $M$ is a real $(2n-1)$-fold. Then we get a map $\int_M : H^{n,n-1}(X) \rightarrow \mathbb{C}$, so $M$ gives a point in $H^{n,n-1}(X)^*$.

If we choose a different $M'$ with $D = \partial M'$, then $\partial(M - M') = 0$ so $M - M' \in H_{2n-1}(X, \mathbb{Z})$. Thus we have a well defined map $[\mathcal{O}(D)] \mapsto [\int_M]$ identifying $\text{Pic}^0$ with $H^{n,n-1}(X)^*/\text{integration over } H_{2n-1}(X, \mathbb{Z})$. 