Three ways to think about vector bundles:
(1) as fiber bundles
(2) in terms of gluing data
(3) as locally free sheaves

1. Vector Bundles as Fiber Bundles

First, we’ll introduce some notation and vocabulary:
Let \( \pi : E \to X \) be a continuous map of topological spaces. Define
\[
E \times_X E = \{(e_1, e_2) \in E \times E \mid \pi(e_1) = \pi(e_2)\}.
\]

A \( \mathbb{C} \)-vector bundle is the data of \( X, E, \pi : E \to X, m : \mathbb{C} \times E \to E \), and \( p : E \times_X E \to E \) such that there is an open cover \( U_i \) of \( X \) such that over \( U_i \), the tuple
\[
(U_i, \pi^{-1}(U_i), \pi|_{\pi^{-1}(U_i)}, m|_{\mathbb{C} \times \pi^{-1}(U_i)}, p|_{\pi^{-1}(U_i) \times _X \pi^{-1}(U_i)})
\]
is isomorphic to
\[(U_i, U_i \times \mathbb{C}', \text{obvious projection, scalar multiplication, addition}).\]

We also have \( \mathbb{R} \)-vector bundles, defined analogously.

**Example 1.1.** \( X \times \mathbb{R} \) is an \( \mathbb{R} \)-vector bundle.

**Example 1.2.** The Möbius Strip is an \( \mathbb{R} \)-vector bundle–locally it is \( S^1 \times \mathbb{R} \).

**Example 1.3.** If \( X \) is any smooth manifold, \( T_\ast X \) (the tangent bundle) and \( T^\ast X \) (the cotangent bundle) are vector bundles. So are \( \bigwedge^k T^\ast, T^\ast \otimes T_\ast, \text{Sym}^2(T^\ast \oplus T_\ast) \), etc.

Note: \( \oplus, \otimes, \bigwedge^\ast, \text{Sym}^\ast \) can all be identified locally and glued together.

If \( X \) is a complex manifold, then \( (T_\ast)_\mathbb{R}X \) comes with an action \( J \) (corresponding to multiplication by \( i \)) on each fiber. For \( (T_\ast)_\mathbb{R} \), \( J \) acts (functorially) and
\[
T_\ast \otimes \mathbb{C} \cong T_{1,0} \oplus T_{0,1},
\]
where \( T_{1,0} \) is the \( i \) eigenspace, and \( T_{0,1} \) is the \( -i \) eigenspace for \( J \). Sections of \( T_{1,0} \) are \((1,0)\)-vector fields, locally \( \sum f_i \frac{\partial}{\partial z_i} \), where the \( f_i \)'s are smooth, and sections of \( T_{0,1} \) are \((0,1)\)-vector fields. Similarly,
\[
T^\ast \otimes \mathbb{C} \cong T^{1,0} \oplus T^{0,1}.
\]

That is,
\[
(\text{differential 1-forms with } \mathbb{C}-\text{values}) \cong ((1,0) - \text{forms}) \oplus ((0,1) - \text{forms}).
\]

**Definition 1.4.** A **map of vector bundles** is
\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & E' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
X & \xrightarrow{=} & X
\end{array}
\]
such that the above diagram commutes and \( \alpha(e_1 + e_2) = \alpha(e_1) + \alpha(e_2) \) (when defined), and \( \alpha(\lambda e) = \lambda \alpha(e) \), \( \lambda \) a scalar.
2. Vector Bundles in Terms of Gluing Data

Starting with the map $\pi : E \to X$, let $U_i$ be an open cover of $X$ so that we have

$$\phi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C}^r.$$  

For $U_i$ and $U_j$ in our cover, we define the map $\psi_{i\to j}$ to be the composition

$$\psi_{i\to j} : (U_i \cap U_j) \times \mathbb{C}^r \xrightarrow{\phi_i^{-1}} \pi^{-1}(U_i \cap U_j) \xrightarrow{\phi_j^{-1}} (U_i \cap U_j) \times \mathbb{C}^r.$$  

Then $\psi_{i\to j}(u, v) = (u, g_{i\to j}(v))$, where $g_{i\to j}$ is a continuous map from $U_i \cap U_j \to GL_r \mathbb{C}$. On the triple intersection $U_i \cap U_j \cap U_k$, we have

$$g_{k\to j} \cdot g_{j\to i} = g_{k\to i},$$

since

$$\phi_k \circ \phi_j^{-1} \circ \phi_j \circ \phi_i^{-1} = \phi_k \circ \phi_i^{-1}.$$  

Note that it follows that $g_{i\to i} = \text{Id}$ and $g_{i\to j} g_{j\to i} = \text{Id}$.

This brings us to the following theorem.

**Theorem 2.1.** Given a topological space $X$, an open cover $U_i$, and continuous maps $g_{i\to j} : U_i \cap U_j \to GL_r \mathbb{C}$, such that

$$g_{k\to j} \cdot g_{j\to i} = g_{k\to i},$$

there is a unique (up to isomorphism) vector bundle $\pi : E \to X$ giving rise to them.

**Proof (Sketch).** Take $\bigsqcup (U_i \times \mathbb{C}^r)$. For $u \in U_i \cap U_j$, glue $(u, v)$ in $U_i \times \mathbb{C}^r$ to $(u, g_{i\to j}(v))$ in $U_j \times \mathbb{C}^r$. Let $E$ be the result of these gluings, etc. □

A vector bundle is . . .

- smooth if we can arrange that the maps $g_{i\to j} : U_i \cap U_j \to GL_r \mathbb{C}$ are smooth (this makes sense for $X$ a smooth manifold).
- holomorphic if we can arrange that the $g_{i\to j}$ are holomorphic (this makes sense for a complex manifold).
- a local system if we can arrange that the $g_{i\to j}$ are locally constant.

The following is an example of where local systems come from.

**Example 2.2.** Let $p : Y \to X$ be a submersion (i.e. a smooth map). Locally, this looks like $M \times U \to U$ for some $M$. We want to define a vector bundle that is locally $H^q(M, \mathbb{Z}) \times U \to U$. On $M \times (U_i \cap U_j) \to U_i \cap U_j$, glue by

$$g : U_i \cap U_j \to \text{Aut}(M)$$

and

$$g^* : U_i \cap U_j \to GL_r(\mathbb{Z}),$$

where $r$ is the dimension of $H^q(M, \mathbb{Z})$. $g^*$ must be locally constant since the target is discrete! Let $R^q p_*$ be the vector bundle on $X$ whose fibers are $H^q(p^{-1}(x), \mathbb{R})$, glued by $g^*$.

Now, in order to construct a map between vector bundles (over the same base $X$) equipped with gluing data, we need to discuss refinements of their covers.

Given gluing data $U_i$, $g_{i\to j}$, a **refinement** of the cover $U_i$ is:

- An open cover $V_i$, and
- for each $i'$, an index $i$, such that $V_i \subseteq U_i$.

Now, for $V_i$ and $V_{i'}$, with corresponding indices $i$ and $j$, so that $V_i \cap V_{i'} \subset U_i \cap U_{i'}$, let

$$h_{j\to i} = g_{j\to i}|_{V_i \cap V_{i'}}.$$  

This is again gluing data, and gives an isomorphic vector bundle.

Now, let $X$ be a topological space, and let $(U_i, g_{j\to i})$ and $(V_i, h_{j\to i})$ be two sets of gluing data over $X$, where the rank of $U_i$ is $r$ and the rank of $V_i$ is $s$. To give a map from the first to second vector bundle, find a common refinement $W_i$ of $U_i$ and $V_i$. Let our new maps be $g'_{j\to i}$ and $h'_{j\to i}$. On each $W_i$, give a map $a_i : W_i \to \text{Mat}_{s \times r}(\mathbb{C})$ such that $h'_{j\to i} \cdot a_i = a_j \cdot g'_{j\to i}$ on $W_i \cap W_j$. 

The two pairs \((W_i, a_i)\) and \((W'_i, a'_i)\) give the same map of vector bundles if and only if there is a refinement \(W''_i\) where \(a_i|_{W''_i} = a'_i|_{W''_i}\).

A map of smooth vector bundles/holomorphic vector bundles/local systems means that the \(a_i\) are smooth/holomorphic/locally constant.

### 3. Vector Bundles as Locally Free Sheaves

Given a map \(\pi : E \to X\), and \(U \subset X\), let \(\mathcal{E}(U) = \{\sigma : U \to \pi^{-1}(U) \mid \pi \circ \sigma = \text{Id}\}\). \(\mathcal{E}\) is a sheaf, with the restriction maps being the restriction of functions. This is a sheaf of complex vector spaces, with pointwise addition and scalar multiplication. If \(E\) is smooth/holomorphic/a local system, we can define the sheaf of smooth/holomorphic/locally constant sections.

**Warning:** Given a holomorphic vector bundle, we can talk about the sheaf of smooth sections.

**Example 3.1.** \(\mathcal{H}^p\) is the holomorphic sections of \(T^{p,0} = \wedge^p T^{1,0}\) while \(\Omega^{p,0}\) is the smooth sections of \(T^{p,0}\). In general, \(T^{p,q} = \wedge^p T^{1,0} \otimes \wedge^q T^{0,1}\) and \(\Omega^{p,q}\) is smooth sections of \(T^{p,q}\).

**Note:** Not all sheaves are sections of vector bundles. Consider the closed map \(W \hookrightarrow X\). Then \(i_*\mathcal{O}_W\) is not a vector bundle.

- In the topological vector bundle setting, \(\mathcal{E}\) is a continuous \(\mathbb{C}\)-valued functions module. Given a section \(\sigma : U \to \pi^{-1}(U)\), with \(f : U \to \mathbb{C}\), \(f \cdot \sigma\) is also a section.
- Smooth vector bundles give \(C^\infty\)-modules.
- Holomorphic vector bundles give \(\mathcal{O}\)-modules.
- Local systems give \(\text{LC}_\mathbb{C}\)-modules.

A map of \(C^\infty\)-modules \(\mathcal{E} \to \mathcal{F}\) is equivalent to a map of smooth vector bundles \(E \to F\).

**Note:** Consider \(C^\infty \xrightarrow{d \cdot} \Omega^1\). As \(d(f \cdot \sigma) \neq fd(\sigma)\), this is not a map of vector bundles; it is what is called a connection.

**Theorem 3.2** (Swan - Serre). The category of smooth vector bundles/holomorphic vector bundles/local systems on \(X\) is equivalent to the category of locally free \(C^\infty\)-/\(\mathcal{O}\)/LC\(_\mathbb{C}\)-modules.

**What is locally free?** It means that there is an open cover \(U_i\) such that, on \(U_i\), \(\mathcal{E} \cong C^\infty(U_i)^{\oplus r} / \mathcal{O}(U_i)^{\oplus r} / \text{LC}_\mathbb{C}(U_i)^{\oplus r}\).

**Remark 3.3.** In the \(C^\infty\) world, if we know the \(C^\infty(X)\)-module structure of \(\mathcal{E}(X)\), this determines \(\mathcal{E}\) as a sheaf of \(C^\infty\)-modules.

**Proof (Sketch).** Given \(x \in X\), and \(\pi^{-1}(x) \cong \mathcal{E}(X) \otimes_{C^\infty(X)} \mathbb{C}\), where \(\mathbb{C}\) is a \(C^\infty\)-module by “value at \(x\)”. In other words \(\sigma \equiv \sigma'\) if \(\sigma - \sigma' = \sum f_i T_i\), where the \(f_i\) are functions vanishing at \(x\) and the \(T_i\) are global sections.

Locally, if \(\sigma(x) = \sigma'(x)\), then \(\sigma(x) - \sigma'(x) = \sum z_i T_i\), where the \(z_i\) are local coordinates.

If you have a section \(\sigma \in \mathcal{E}(U)\) and \(f \in C^\infty(U)\), find an open cover \(V_i\) of \(U\) such that \(\bar{V}_i\) is compact in \(U\), and a hat function \(T_i\), which is 1 on \(V_i\) and 0 on \(X \setminus U\). Then \(T_i \sigma\) and \(T_i f\) extend to sections in \(\mathcal{E}(X)\) and \(C^\infty(X)\). So on \(V_i\), \(f \cdot \sigma = (T_i f)(T_i \sigma)\). So we know what \(f \sigma|_{V_i}\) is, so we know what \(f \sigma\) is.  

\(\square\)