Worksheet 12: Products of Chevalley generators in $GL_n$

We write $x_j(t)$ for the matrix which has $t$ in position $(j, j+1)$, which has 1’s on the diagonal and 0’s everywhere else. We write $y_j(t)$ for the matrix which has $t$ in position $(j+1, j)$, which has 1’s on the diagonal and 0’s everywhere else. We write $\delta_j(t)$ for the diagonal matrix which is $t$ in position $(j, j)$ and has 1’s in the other diagonal places.

We earlier proved:

\[
x_i(\mathbb{R}_0) x_j(\mathbb{R}_0) = x_j(\mathbb{R}_0) x_i(\mathbb{R}_0) \quad \text{for} \quad |i - j| \geq 2,
\]
\[
x_i(\mathbb{R}_0) x_{i+1}(\mathbb{R}_0) x_i(\mathbb{R}_0) = x_{i+1}(\mathbb{R}_0) x_i(\mathbb{R}_0) x_{i+1}(\mathbb{R}_0).
\]

Of course, we also have

\[
y_i(\mathbb{R}_0) y_j(\mathbb{R}_0) = y_j(\mathbb{R}_0) y_i(\mathbb{R}_0) \quad \text{for} \quad |i - j| \geq 2.
\]
\[
y_i(\mathbb{R}_0) y_{i+1}(\mathbb{R}_0) y_i(\mathbb{R}_0) = y_{i+1}(\mathbb{R}_0) y_i(\mathbb{R}_0) y_{i+1}(\mathbb{R}_0).
\]
\[
y_i(\mathbb{R}_0) y_i(\mathbb{R}_0) = y_i(\mathbb{R}_0).
\]

**Problem 12.1.** Show that

\[
\delta_i(\mathbb{R}_0) \delta_j(\mathbb{R}_0) = \delta_j(\mathbb{R}_0) \delta_i(\mathbb{R}_0)
\]
\[
\delta_i(\mathbb{R}_0) x_j(\mathbb{R}_0) = x_j(\mathbb{R}_0) \delta_i(\mathbb{R}_0)
\]
\[
\delta_i(\mathbb{R}_0) y_j(\mathbb{R}_0) = y_j(\mathbb{R}_0) \delta_i(\mathbb{R}_0)
\]
\[
\delta_i(\mathbb{R}_0) \delta_i(\mathbb{R}_0) = \delta_i(\mathbb{R}_0).
\]

for $i \neq j$.

**Problem 12.2.** Show that

\[
x_i(\mathbb{R}_0) y_j(\mathbb{R}_0) = y_j(\mathbb{R}_0) x_i(\mathbb{R}_0)
\]

**Problem 12.3.** Show that

\[
x_i(\mathbb{R}_0) y_i(\mathbb{R}_0) \delta_i(\mathbb{R}_0) \delta_{i+1}(\mathbb{R}_0) = y_i(\mathbb{R}_0) x_i(\mathbb{R}_0) \delta_i(\mathbb{R}_0) \delta_{i+1}(\mathbb{R}_0).
\]

**Problem 12.4.** Consider any product where each term is of the form $x_i(\mathbb{R}_0)$, $y_j(\mathbb{R}_0)$, $\delta_k(\mathbb{R}_0)$ and where each of $\delta_1(\mathbb{R}_0)$, $\delta_2(\mathbb{R}_0)$, ..., $\delta_n(\mathbb{R}_0)$ appears at least once. Let $i_1$, $i_2$, ..., $i_M$ be the sequence of subscripts of the $x_i$ factors and let $j_1$, $j_2$, ..., $j_N$ be the sequence of subscripts of the $y_j$ factors. Prove that the image of this product in $GL_n(\mathbb{R})$ depends only on the 0-Hecke products $e_{i_1} * e_{i_2} * ... * e_{i_M}$ and $e_{j_1} * e_{j_2} * ... * e_{j_N}$.

Let $u$ and $v$ in $S_n$. We define $M^{u,v}$ to be the product in Problem 12.4, where $u = e_{i_1} * e_{i_2} * ... * e_{i_M}$ and $v = e_{j_1} * e_{j_2} * ... * e_{j_N}$.

**Problem 12.5.** Show that $M^{u,v} \subseteq B_- u B_- \cap B_+ v B_+$.

As you would guess, our eventual goal is that $M^{u,v}$ is the totally nonnegative part of $B_- u B_- \cap B_+ v B_+$ and, if $s_{i_1} \cdots s_{i_M}$ and $s_{j_1} \cdots s_{j_N}$ are reduced, then this gives a diffeomorphism $M^{u,v} \cong \mathbb{R}_0^{n+\ell(u)+\ell(v)}$. This is a theorem of Fomin and Zelevinsky, “Double Bruhat Cells and Total Positivity”, *JAMS*, Volume 12, Number 2, April 1999, Pages 335–380.

This is a good chance to prove a lemma which we’ll need in the future:

**Problem 12.6.** Show that there is a continuous (in fact, polynomial) function $g : \mathbb{R}_0 \to GL_n(\mathbb{R})$ such that $g(t)$ is totally positive for $t > 0$ and $g(0) = Id_n$. 