We recall the reader of the following notational conventions: \([k] = \{1, 2, \ldots, k\}\). The minor \(\Delta^R_C(X)\) uses row set \(R\) and column set \(C\).

An **LDU factorization** of an \(n \times n\) matrix \(X = LDU\) where \(L \in N_-\), \(U \in N_+\) and \(D\) is diagonal. In this course, we will always require that \(D\) is diagonal and **invertible**.

**Problem 13.1.** Show that, if \(X\) has an **LDU** factorization, then the minors \(\Delta^k_k(X)\) are nonzero for \(1 \leq k \leq n\).

**Problem 13.2.** Show the converse of Problem 13.1: If \(X\) is an \(n \times n\) matrix and \(\Delta^k_k(X) \neq 0\) for \(1 \leq k \leq n\), then \(X\) has an **LDU**-factorization.

**Problem 13.3.** Let \(X = LDU\) be an **LDU** factorization. Show that the entries of \(L\), \(D\) and \(U\) are given by the following formulas:

\[
L_{ij} = \frac{\Delta_{[j]}^{[j-1]\cup\{i\}}(X)}{\Delta_{[j]}^{[j]}(X)} \quad \text{for} \quad i > j \quad U_{ij} = \frac{\Delta_{[k]}^{[k-1]\cup\{j\}}(X)}{\Delta_{[k]}^{[k]}(X)} \quad \text{for} \quad i < j \quad D_{jj} = \frac{\Delta_{[j]}^{[j]}(X)}{\Delta_{[j-1]}^{[j-1]}(X)}.
\]

In particular, deduce that \(L\), \(D\) and \(U\) are uniquely determined by \(X\).