We define the following subgroups of $\text{GL}_n$:

$$B_+ = \begin{bmatrix}
* & * & * & \cdots & * \\
* & * & * & \cdots & * \\
* & \cdots & * \\
\vdots & \vdots & \vdots \\
* 
\end{bmatrix}, \quad B_- = \begin{bmatrix}
* \\
* \\
* \\
\vdots \\
* 
\end{bmatrix}, \quad B_{+}^{(n)} \quad \text{etcetera when the size of the matrix is not clear from context.}

Last time we stated the Borel decomposition of $\text{GL}_n$:

$$\text{GL}_n = \bigsqcup_{w \in S_n} B_- w B_+.$$  

And we stated that, more explicitly: $B_- w B_+$ is the set of matrices where the ranks of all upper left submatrices match the rank of $w$.

We will want to prove this and, for inductive purposes, it will be better to prove something stronger. Define a partial permutation matrix to be a $(0, 1)$-matrix $\pi$ where each row and column contains at most one 1. For example, here is a complete list of all $2 \times 2$ partial permutation matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  

We’ll write $\text{PP}_{mn}$ for the set of $m \times n$ partial permutation matrices.

**Problem 6.1.** Show that the ranks of upper left submatrices are unchanged by left multiplication by $B_-$ and right multiplication by $B_+$.

**Problem 6.2.** Let $X$ be an $m \times n$ matrix. Show that there is a unique $m \times n$ partial permutation matrix $\pi$ such that the ranks of each upper left submatrix match those of $\pi$.

**Problem 6.3.** Let $X$ be an $m \times n$ matrix and let $\pi$ be the partial permutation matrix whose upper left ranks match those of $X$. Show that there are matrices $b_- \in B_-^{(m)}$ and $b_+ \in B_+^{(n)}$ such that $X = b_- \pi b_+$. 
