The representation of $X$ as $b_+ wb_+$ is generally not unique. We now discuss getting a unique representation. For simplicity, we switch back to the case of permutation matrices.

Here is where we are going:

**Theorem.** Each matrix in $B_- w B_+$ has a unique factorization in the forms

$$B_- w(w^{-1} N_+ w \cap N_+) = (N_- \cap w N_- w^{-1})(B_- w \cap w B_+)(w^{-1} N_+ w \cap N_+) = (N_- \cap w N_- w^{-1})w B_+.$$  

Let $\Phi_+ = \{(i, j) : 1 \leq i < j \leq n\}$. For any $X \subseteq \Phi_+$, let $N_+(X)$ be $\{g \in N_+ : g_{ij} = 0 \text{ for } (i, j) \notin X\}$.

**Problem 7.1.** Show that $N_+ \cap w N_+ w^{-1}$ is $N_+(X)$ for a certain set $X$, and describe $X$ explicitly. Show that $\#(X) = \binom{n}{2} - \ell(w)$.

**Problem 7.2.** For any subset $X$ of $\Phi_+$, show that every element of $N_+$ has a unique factorization of the form $N_+(X) N_+(\Phi_+ \setminus X)$.

**Problem 7.3.** Show that every element of $N_+$ has a unique factorization in the form

$$(N_+ \cap w^{-1} N_- w)(N_+ \cap w^{-1} N_+ w).$$

**Problem 7.4.** Show that every element of $B_+$ has a unique factorization in the form

$$(B_+ \cap w^{-1} B_- w)(N_+ \cap w^{-1} N_+ w).$$

**Problem 7.5.** Show that, if any of the unique factorization claims in the theorem is true, then they all are true.

**Problem 7.6.** Show that every matrix in $B_- w B_+$ has at least one factorization as in the theorem.

**Problem 7.7.** Show that every matrix in $B_- w B_+$ has at most one factorization as in the theorem. Hint: You’ll want to work with one of the forms with two factors.

One might want variants of this theorem for $B_{\pm 1} w B_{\pm 2}$ for any of the four sign choices. Here is the correct statement:

**Theorem.** Let $\pm_1$ and $\pm_2$ be two choices of $+$ or $-$. Then every matrix in $B_{\pm_1} w B_{\pm_2}$ has a unique factorization in any of the forms

$$B_{\pm_1} w(w^{-1} N_{\pm_1} w \cap N_{\pm_2}) = (N_{\pm_1} \cap w N_{\pm_2} w^{-1})(B_{\pm_1} w \cap w B_{\pm_2})(w^{-1} N_{\pm_1} w \cap N_{\pm_2}) = (N_{\pm_1} \cap w N_{\pm_2} w^{-1})w B_{\pm_2}.$$  

The intersections $N_{\pm_1} \cap w N_{\pm_2} w^{-1}$ and $w^{-1} N_{\pm_1} w \cap N_{\pm_2}$ are isomorphic as manifolds to $\mathbb{R}^{\ell(w)}$ if $\pm_1 = \pm_2$ and to $\mathbb{R}^{\binom{n}{2} - \ell(w)}$ if $\pm_1 = \mp_2$.

**Remark.** The intersection $N_+ \cap w N_+ w^{-1}$ is a Lie group. The corresponding Lie algebra is $n_+ \cap w n_+ w^{-1}$, where $n_+$ is the upper triangular matrices with zeroes on the diagonal. The exponential map $n_+ \cap w n_+ w^{-1} \rightarrow N_+ \cap w N_+ w^{-1}$ is an isomorphism. This is the conceptually right reason that $N_+ \cap w N_+ w^{-1} \cong \mathbb{R}^{\binom{n}{2} - \ell(w)}$.

**Remark.** The factorization $N_+ = N_+(X) N_+(\Phi_+ \setminus X)$ is correct for any subset $X$ of $\Phi_+$, but $N_+(X)$ and $N_+(\Phi_+ \setminus X)$ are not always groups. In fact, the sets $X$ for which $N_+(X)$ and $N_+(\Phi_+ \setminus X)$ are both groups are precisely those arising from permutations $w$. 