THE FROBENIUS CHARACTER MAP

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Let \( d \leq n \). Then Schur-Weyl duality gives us an equivalence of categories:

\[
\{\text{Finite dimensional } S_d \text{ representations}\} \rightarrow \{\text{Polynomial } GL_n \text{ representations where } t \cdot \text{Id acts by } t^d \}
\]

Every element of the left hand category has a character, which is a class function on \( S_d \). Let \( \Lambda^d \) denote the vector space of symmetric polynomials of degree \( d \). To every representation in the right hand category, taking the character assigns an element \( \Lambda^d \). So we have a linear map

\[
F : \{\text{Class functions on } S_d\} \rightarrow \Lambda^d.
\]

This map is called the Frobenius character map, and it is the goal of this note to describe it. For a permutation \( \sigma \in S_d \), let \( c(\sigma) \) be the partition whose parts are the lengths of the cycles of \( \sigma \). For example, the identity permutation maps to \( 1^d \); a simple transposition maps to \( 2 1^{d-1} \).

1. The \( h, e \) and \( s \) bases

We can figure out some values of \( F \) by looking at the inverse correspondence when \( n = d \).

1.1. The \( h \) basis. We know that \( h_\lambda \) is the character of the representation \( \bigotimes_k \text{Sym}^{\lambda_k}(V) \), which I’ll abbreviate \( H \). I’ll abbreviate the \( (1,1,1,1,1,1) \) weight space by \( H_0 \).

For concreteness sake, take \( \lambda = (4,2,1) \). Then \( H_0 \) has as basis elements of the form

\[
(z_{i_1} z_{i_2} z_{i_3} z_{i_4}) \otimes (z_{j_1} z_{j_2}) \otimes z_k
\]

where \( \{1,2,3,4,5,6,7\} \) is the disjoint union of \( \{i_1,i_2,i_3,i_4\}, \{j_1,j_2\} \) and \( \{k\} \). So a basis for \( H_0 \) can be indexed by partitions of \( d \) into sets of size \( \lambda_1, \lambda_2, \ldots, \lambda_r \). The symmetric group acts by permuting those set partitions.

If a group \( G \) acts by permuting a finite set \( X \), then the character of the permutation representation \( \mathbb{C}X \) is

\[
\chi_{\mathbb{C}X}(g) = \#(X^g)
\]

In our case, we come to the following conclusion: Let \( \mu = c(\sigma) \). Then \( \chi_{H_0}(\sigma) \) is the number of ways to partition the multiset \( \{\mu_1,\mu_2,\ldots,\mu_s\} \) into multisubsets whose sizes are \( \lambda_1, \lambda_2, \ldots, \lambda_r \). For example, suppose that \( \lambda = (4,2,1) \) and \( c(\sigma) = (2,2,1,1,1,1) \). Then \( \chi_{H_0}(\sigma) = 9 \), corresponding to

\[
\begin{align*}
(2+2,1+1,1) & \quad (2+2,1+1,1) & \quad (2+2,1+1,1) & \quad (2+1+1,2,1) & \quad (2+1+1,2,1) \\
(2+1+1,2,1) & \quad (2+1+1,2,1) & \quad (2+1+1,2,1) & \quad (2+1+1,2,1) & \quad (2+1+1,2,1)
\end{align*}
\]

The coloring is meant to make it clear that we must keep track of the distinct identities of the two 2’s and the three 1’s.

So the character of \( H_0 \) is the class function

\[
\sigma \mapsto \#\{\text{set partitions of } c(\sigma) \text{ with parts of size } (\lambda_1,\lambda_2,\ldots,\lambda_r)\},
\]

and \( F \) maps it to \( h_\lambda \).

1.2. The \( e \) basis. Similarly, we know that \( e_\lambda \) is the character of the representation \( \bigotimes_k \text{Sym}^{\lambda_k}(V) \). Let \( E = \bigotimes_k \text{Sym}^{\lambda_k}(V) \) and let \( E_0 \) be the \( (1,1,\ldots,1) \) weight space. Taking \( \lambda = (4,2,1) \) again, \( E_0 \) has as a basis the products of minors

\[
\Delta_{i_1 i_2 i_3 i_4} \Delta_{j_1 j_2} \Delta_k.
\]

Once again, our basis is indexed by set-partitions of \( \{1,2,\ldots,n\} \) into sets of size \( \lambda_1, \lambda_2, \ldots, \lambda_r \), but there is a sign factor. We deduce that

\[
E_0 \cong H_0 \otimes \text{sign}.
\]
The character of $E_0$ is the class function
\[
\sigma \mapsto (-1)^{\sigma} \cdot \#\{\text{set partitions of } c(\sigma) \text{ with parts of size } (\lambda_1, \lambda_2, \ldots, \lambda_r)\},
\]
We deduce that this character maps to $e_1$.

This would have perhaps been a slicker proof that $\omega$ corresponds to tensor with sign, since we know that $\omega(h_\lambda) = e_\lambda$.

1.3. The $s$ basis. We defined the Specht module $Sp(\lambda)$ to be the $(1, 1, \ldots, 1)$ weight space of $V_\lambda(d)$. So the character of the Specht module maps to $s_\lambda$.

2. A GENERAL FORMULA

Let $f : S_d \to \mathbb{C}$ be a class function. We claim that
\[
F(f) = \frac{1}{d!} \sum_{\sigma \in S_d} f(\sigma) \text{Tr}(\sigma \times \text{diag}(t_1, t_2, \ldots, t_n))
\]
Here $\sigma \times \text{diag}(t_1, t_2, \ldots, t_n)$ is an element of $S_d \times GL_n$, and we are considering its action on $V^\otimes d$ where $V = \mathbb{C}^n$.

It is enough to prove this result for $f$ the character of an $S_d$-irrep. Let $\chi_\lambda$ be the character of $Sp(\lambda)$. Set
\[
\pi_{Sp(\lambda)} = \frac{\dim Sp(\lambda)}{d!} \sum_{\sigma \in S_d} \chi_\lambda(\sigma) \rho_{Sp(\lambda)}(\sigma),
\]
an element in $\mathbb{C}[S_d]$. From the October 3 lecture, $\pi_{Sp(\lambda)}$ acts by 1 on $Sp(\lambda)$ and acts by 0 on $Sp(\mu)$ for $\mu \neq \lambda$. We have
\[
\frac{1}{d!} \sum_{\sigma \in S_d} \chi_{Sp(\lambda)}(\sigma) \sigma \times \text{diag}(t_1, t_2, \ldots, t_n) = \frac{1}{\dim Sp(\lambda)} \pi_{Sp(\lambda)} \otimes \text{diag}(t_1, t_2, \ldots, t_n).
\]
By Schur-Weyl duality, the trace of the above operator on $V^\otimes d$ is $s_\lambda(t_1, t_2, \ldots, t_n)$, since it acts by 0 on $Sp(\mu) \otimes V_\mu(n)$ for $\mu \neq \lambda$ and acts by $\frac{1}{\dim Sp(\lambda)} \times \text{diag}(t_1, \ldots, t_n)$ on $Sp(\lambda) \otimes V_\lambda(n)$. \(\square\)

3. THE POWER SYMMETRIC FUNCTIONS

We can now see where the power symmetric functions come from. Let us extend $F$ to functions on $S_d$ which are not class functions, using formula (1). Let $\sigma$ be an element of $S_d$ with $c_\sigma = \mu$, and let $\delta_{\sigma}$ be the function which is 1 on $\sigma$ and 0 elsewhere. So
\[
F(\delta_{\sigma}) = \frac{1}{d!} \text{Tr}(\sigma \times \text{diag}(t_1, \ldots, t_n))
\]
Consider the action of $\sigma \times \text{diag}(t_1, \ldots, t_n)$ on the obvious basis $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}$ of $V^\otimes d$. Let $(a_1, a_2, \ldots, a_j)$ be one of the orbits of $\sigma$. If we are to map $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}$ to a multiple of itself, we must have $i_{a_1} = i_{a_2} = \cdots = i_{a_d}$.

So the nonzero contributors to $F(\delta_{\sigma})$ are indexed by functions $\{\text{orbits of } \sigma\} \to \{1, \ldots, n\}$. Given an orbit $\Omega$ of $\sigma$, if it is mapped to $k$, we see that we get a contribution of $t_k^{\mid \Omega \mid}$. So
\[
F(\delta_{\sigma}) = \frac{1}{d!} \sum_{\phi: \text{orbits(}\sigma\text{)} \to \{1, \ldots, n\}} \prod_{\Omega \in \text{orbits}(\sigma)} t_{\phi(\Omega)}^{\mid \Omega \mid}.
\]
We can factor the sum as
\[
\prod_{\Omega \in \text{orbits}(\sigma)} \left(t_1^{\mid \Omega \mid} + t_2^{\mid \Omega \mid} + \cdots + t_n^{\mid \Omega \mid}\right)
\]
which is
\[
\prod_{k=1}^{s} (t_1^{\mu_k} + t_2^{\mu_k} + \cdots + t_n^{\mu_k}) = p_\mu(t).
\]
In summary,
\[
F(\delta_{\sigma}) = \frac{1}{d!} p_\mu.
\]
3.1. **Orthogonality of the power symmetric functions.** Let $M(\mu)$ be the size of the conjugacy class of $\sigma$, with $\mu = c(\sigma)$. Let $\epsilon_\mu$ be the function which is 1 on elements with cycle structure $\mu$, and 0 on all other elements. By linearity

\[ F(\epsilon_\mu) = \frac{M(\mu)}{d!} p_\mu. \]

Let $(\ , \ )$ be the inner product $\frac{1}{d!} \sum_{\sigma \in S_d} f(\sigma) \overline{g(\sigma)}$ on class functions of $S_d$. Character functions of $S_d$-irreps are orthonormal for $(\ , \ )$, and $F$ sends characters of irreps to Schurs, so $(f, g) = (F(f), F(g))$. In particular, for $\lambda \neq \mu$, we have

\[ \langle p_\lambda, p_\mu \rangle = \text{constant} \cdot (\epsilon_\lambda, \epsilon_\mu). \]

But the right hand side is clearly zero, since $\epsilon_\lambda$ and $\epsilon_\mu$ have distinct supports in $S_d$. So we now have a conceptual explanation for why the power symmetrics are orthogonal.

We can also deduce something interesting by pairing $\epsilon_\mu$ with itself. On the one hand

\[ (\epsilon_\mu, \epsilon_\mu) = \frac{M(\mu)^2}{(d!)^2 z_\mu} \]

in the notation of Problem Set 1. On the other hand, it is clear that

\[ (\epsilon_\mu, \epsilon_\mu) = \frac{M(\mu)}{d!}. \]

So

\[ \frac{d!}{M(\mu)} = \frac{1}{z_\mu}. \]

Notice that $d!/M(\mu)$ is the size of the centralizer, $Z(\sigma)$, of $\sigma$. So $z_\mu = 1/|Z(\sigma)|$. (Some of you pointed out that Stanley defines $z_\mu$ to be the reciprocal of what I wrote; this convinces me his definition is better.)

4. **The Frobenius character formula**

Let $\chi_\lambda$ be the character of $Sp(\lambda)$. So we have

\[ \chi_\lambda = \sum_{\sigma} \chi_\lambda(\sigma) \delta_{\sigma} \]

as functions on $S_d$. Applying $F$ to both sides, we deduce

\[ s_\lambda = \sum_{\sigma \in S_d} \chi_\lambda(\sigma) \frac{p_c(\sigma)}{d!} = \sum_{|\mu| = d} \chi_\lambda(\mu) \frac{M(\mu)}{d!} p_\mu. \]

In other words, the character table of $S_d$ is, up to some minor conversion factors, the change of basis matrix from power symmetric’s to Schurs.

In fact, we can make it look nicer by switching the roles of $p$ and $s$. Using the self-orthogonality of $p$’s and $s$’s, we can compute

\[ \chi_\lambda(\mu) = \langle s_\lambda, p_\mu \rangle \]

\[ p_\mu = \sum_{\lambda} \chi_\lambda(\mu) s_\lambda \]

On Problem Set 8, you’ll derive a combinatorial formula for $\langle s_\lambda, p_\mu \rangle$. 