NOTES FOR NOVEMBER 19, 2012: INTERACTION BETWEEN \( \mathfrak{gl}_2 \) STRINGS

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1. FROM LAST CLASS (\( \mathfrak{gl}_2 \) REPRESENTATIONS)

The Lie algebra \( \mathfrak{gl}_n \) consists of the \( n \times n \) matrices with bracket \([A,B] = AB - BA\). It has a basis of elementary matrices \( E_{ij} \) with a single 1 in \( i \)th row and \( j \)th column, and 0’s everywhere else.

Define

\[
h_k = E_{kk}, \quad e_k = E_{k(k+1)}, \quad f_k = E_{(k+1)k}.\]

In particular,

\[
h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

are elements of \( \mathfrak{gl}_2 \).

For any smooth representation \( \rho : \text{GL}_n \to \text{GL}(V) \), the differential of \( \rho \) is a Lie algebra representation \( \sigma : \mathfrak{gl}_n \to \mathfrak{gl}(V) = \text{End}(V) \). If \( v \in V \) is in the \((p_1, \ldots, p_n)\) weight space, that is, if

\[
\rho \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \cdot v = (t_1^{p_1} \cdots t_n^{p_n})v,
\]

then \( \sigma(E_{ij})v \) is in the \((p_1, p_2, \ldots, p_i + 1, \ldots, p_j - 1, \ldots, p_n)\) weight space.

Consider the \( \text{GL}_2 \)-representation \( V(k)(2) \) and the corresponding \( \mathfrak{gl}_2 \)-representation. It has a weight basis with weights \((k, 0), (k-1, 1), \ldots, (0, k)\). In that (ordered) basis, the \( \mathfrak{gl}_2 \)-representation is given by

\[
\sigma : \mathfrak{gl}_2 \to \text{End}(V(k)(2)) \quad h_1 \mapsto \begin{pmatrix} k & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad h_2 \mapsto \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},
\]

\[
e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ \vdots & \ddots \end{pmatrix}, \quad f_1 \mapsto \begin{pmatrix} 0 & \cdots & 0 \\ k & \cdots & 0 \\ \vdots & 2 & 0 \end{pmatrix}.
\]

The actions of \( e_1 \) and \( f_1 \) on \( V(k)(2) \) are captured in the following picture

\[
(0, k) \quad \bullet \quad k \quad \bullet \quad k - 1 \quad \bullet \quad 1 \quad \bullet \quad 2 \quad \bullet \quad \ldots \quad \bullet \quad (k, 0) 
\]

\[ (0, k) \xrightarrow{e_1} (k, 0) \quad \bullet \quad (k-1, 1) \quad \bullet \quad (k-2, 2) \quad \ldots \quad \bullet \quad (1, k-1) \quad \bullet \quad (0, k) 
\]

\[ k \xrightarrow{f_1} k - 1 \quad \bullet \quad (k, 0) \quad \bullet \quad (k-1, 1) \quad \bullet \quad (k-2, 2) \quad \ldots \quad \bullet \quad (1, k-1) \quad \bullet \quad (0, k) 
\]
where the dots (●) are weight spaces and the arrows for \( e_1 \) and \( f_1 \) are labeled with their factor.

For the representation \( V_{(k,\ell)}(2) \cong (\det)^{\otimes \ell} \otimes V_{(k-\ell)} \), the picture for \( V_{(k-\ell)} \) is shifted to the weights \((k,\ell), (k-1,\ell+1), \ldots, (\ell,k)\).

\[
\begin{array}{c}
(\ell,k) \\
\vdots \\
1 \\
2 \\
\vdots \\
k-\ell \\
1 \\
(k,\ell)
\end{array}
\]

**Remark:** \( \mathfrak{sl}_2 \) controls the world!

### 2. Interaction of \( \mathfrak{gl}_2 \) Strings

How do \((e_j, f_j)\) act on \((e_k, f_k)\) strings of a \( \mathfrak{gl}_n \)-representation?

**Proposition 1.** For any \( \mathfrak{gl}_n \)-representation:
- If \(|j-k| \geq 2\), then \((e_j, f_j)\) preserves the length of the \((e_k, f_k)\) string.
- If \(|j-k| = 1\), then \(e_j\) and \(f_j\) can only map between strings whose lengths differ by \(\pm 1\).

**Proof.** Let \((\sigma, W)\) be a \( \mathfrak{gl}_n \)-representation. Define

\[
\phi_i : \mathfrak{gl}_2 \rightarrow \mathfrak{gl}_n
\]

by inserting the \(2 \times 2\) matrix in the rows \(i\) and \(i+1\) and the columns \(i\) and \(i+1\), with everything else zero.

\[
\begin{pmatrix}
0 & \ddots & \ & \ \\
& \ & \ast & \ast \\
& \ & \ast & \ast \\
& \ & \ddots & \\
& \ & \ & 0
\end{pmatrix}
\]

Suppose that \(|j-k| \geq 2\). Then \([\phi_j(\mathfrak{gl}_2), \phi_k(\mathfrak{gl}_2)] = 0\). Any two matrices \(u \in \phi_j(\mathfrak{gl}_2)\) and \(v \in \phi_k(\mathfrak{gl}_2)\) must commute, so \(\sigma(u) : W \rightarrow W\) is a map of \(\phi_k(\mathfrak{gl}_2)\)-representations. So the \(\phi_k(\mathfrak{gl}_2)\)-isotypic components of \(W\) are invariant under \(\sigma(u)\), and \(\sigma(u)\) preserves the length of \((e_k, f_k)\) strings.

The proof of the second statement uses the Serre relation, which will be introduced later. \(\square\)

**Example 2.** Consider \( V_{(2,1)}(3) \). It has weight spaces

\[
\begin{array}{c}
(1,2,0) \\
(2,1,0) \\
(0,2,1) \\
(1,1,1) \\
(2,0,1) \\
(0,1,2) \\
(1,0,2)
\end{array}
\]

and all of the weight spaces are one-dimensional except for \((1,1,1)\), which is two-dimensional.
The $(1, 1, 1)$ weight space contains the semistandard basis elements \( \{ \Delta_{12}\Delta_3, \Delta_{13}\Delta_2 \} \). If we instead split the $(1, 1, 1)$ weight space into the basis \( \{ \frac{1}{2}(\Delta_{23}\Delta_1 + \Delta_{13}\Delta_2), \frac{1}{2}(\Delta_{23}\Delta_1 - \Delta_{13}\Delta_2) \} \), we have the picture

where the horizontal solid arrows are \((e_1, f_1)\) strings and the diagonal dashed arrows are \((e_2, f_2)\) strings.

The \((e_1, f_1)\) and \((e_2, f_2)\) strings come from the restriction of the \(GL_3\)-representation \( V_{(2,1)}(3) \) to the subgroups

\[
\begin{pmatrix}
* & * & 0 \\
* & * & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & * & * \\
0 & * & *
\end{pmatrix}
\]

both isomorphic to \(GL_2\).

For an example of the computations involved, consider the \(f_2\) string

\( \Delta_{12}\Delta_2 \xrightarrow{f_2} \frac{1}{2}(\Delta_{23}\Delta_1 + \Delta_{13}\Delta_2) \xrightarrow{f_2} \Delta_{13}\Delta_3. \)

Act on \(\Delta_{12}\Delta_2\) with the \(GL_3\) action by \((1 + \epsilon f_2)\)

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \epsilon & 1
\end{pmatrix} \cdot \begin{pmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
z_{12} & z_{12}
\end{pmatrix} = \begin{pmatrix}
z_{11} & z_{12} + \epsilon z_{13} \\
z_{21} & z_{22} + \epsilon z_{23}
z_{12} & z_{12} + \epsilon z_{13}
\end{pmatrix}
\]

\[= \Delta_{12}\Delta_2 + \epsilon(\Delta_{12}\Delta_3 + \Delta_{13}\Delta_2)\]
so that we get $\Delta_{12}\Delta_3 + \Delta_{13}\Delta_2$. The action of the Lie algebra also is a derivation (i.e. satisfies the Liebniz rule), so we can compute the same thing with

$$f_2(\Delta_{12}\Delta_2) = \Delta_{12}f_2(\Delta_2) + f_2(\Delta_{12})\Delta_2 = \Delta_{12}\Delta_3 + \Delta_{13}\Delta_2.$$ 

Note that it has the correct factor of 2 times the basis element.

The second segment of the string is

$$f_2\left(\frac{1}{2}(\Delta_{12}\Delta_3 + \Delta_{13}\Delta_2)\right) = \frac{1}{2}(f_2(\Delta_{12})\Delta_3 + \Delta_{12}f_2(\Delta_3) + f_2(\Delta_{13})\Delta_2 + \Delta_{13}f_2(\Delta_2))$$

$$= \frac{1}{2}(\Delta_{13}\Delta_3 + \Delta_{13}\Delta_3)$$

$$= \Delta_{13}\Delta_3$$

as claimed.

From the pictures, one can also verify that the actions of $(e_1, f_1)$ change the string length of $(e_2, f_2)$ by $\pm 1$, and vice versa.

### 3. Serre relation

**Proposition 3** (Serre relation). For any $\mathfrak{gl}_n$-representation $\sigma : \mathfrak{gl}_n \to \text{End}(W)$, we have

$$\sigma(f_k)^2\sigma(f_{k+1}) - 2\sigma(f_k)\sigma(f_{k+1})\sigma(f_k) + \sigma(f_{k+1})\sigma(f_k)^2 = 0$$

et cetera. Similar relations hold with $f_k$ and $f_{k+1}$ swapped, and with $e$'s turned into $f$'s.

For notation simplicity, we drop the $\sigma$'s.

**Proof.** For the first equations,

$$f_k^2 f_{k+1} - 2f_k f_{k+1}f_k + f_{k+1} f_k^2 = [f_k, f_k f_{k+1} - f_{k+1} f_k]$$

$$= [f_k, [f_k, f_{k+1}]]$$

$$= [E_{(k+1)k}, [E_{(k+1)k}, E_{(k+2)(k+1)}]]$$

$$= [E_{(k+1)k}, -E_{(k+2)k}] = 0.$$ 

The other equations are similar, and the map of Lie algebras preserves the bracket.  

The Serre relation can be used to show that two $(e_k, f_k)$ strings whose lengths differ by something other than $\pm 1$ cannot be mapped to each other by $f_{k+1}$ (or by $e_{k+1}, f_{k-1},$ or $e_{k-1}$).

**Proof.** Suppose that the target string is at least two longer than the source string. Then at least one end has two extra nodes. Consider this illustration, where left arrows ($\leftarrow$) are $f_k$, right arrows ($\rightarrow$) are $e_k$, and dashed diagonal arrows ($\triangleleft\triangleright$) are $f_{k+1}$.

```
  u0   u1   u2   u3
    \--\    \--\    \--\    \--\
    \--\    \--\    \--\    \--\
  v6   v5   v4   v3   v2   v1   v0
```

In this picture, in order to get the Serre relation

$$f_k^2 f_{k+1} - 2f_k f_{k+1}f_k + f_{k+1} f_k^2 = 0$$

starting from the vector $u_0$, we must have $f_{k+1} \cdot u_0$. Similarly, $f_{k+1} \cdot u_1 = 0$, and by induction $f_{k+1} u_i = 0$ for all $u_i$ in the top string.

If the target string is at least two shorter than the source string, then we have this picture.

```
  v6   v5   v4   v3   v2   v1   v0
    \--\    \--\    \--\    \--\    \--\    \--\    \--\
    \--\    \--\    \--\    \--\    \--\    \--\    \--\
```

For the Serre relation to hold here, $f_{k+1}(f_k^2(v_0)) = 0$. Again by induction, all $f_{k+1}$ maps are zero.

If the two strings are the same length, the $f_{k+1}$ maps cannot go between the relevant weights.
The above arguments also work for $f_{k+1}, e_{k+1}, f_{k-1}$, and $e_{k-1}$. Thus, these maps can only map $(e_k, f_k)$ strings to each other if the lengths of the strings differ by $\pm 1$. \qed