We continue our study of $GL_n$ representations and begin connecting them to unitary representations.

1. Setup

We will use the following notation throughout:

$$G = GL_n(\mathbb{C}),$$
$$K = U_n(\mathbb{C}) = \{ U \in GL_n(\mathbb{C}) : U^T U = \text{Id} \} = \text{the unitary group},$$
$$T = \{ \text{diagonal matrices in } GL_n(\mathbb{C}) \} = \text{(the torus)},$$
$$S = K \cap T.$$  

Note that $K$, the unitary group, is compact, as is $S$. Our goal is to get from understanding the unitary group to understanding $GL_n$. First, we need some facts from complex analysis.

**Lemma 1.** Let $f$ be an analytic function defined on an open neighborhood $U$ of 0 in $\mathbb{C}^n$. If $f \equiv 0$ on $\mathbb{R}^n \cap U$, then $f = 0$.

**Proof.** By induction: the base case is clear ($\mathbb{C}^0 = \mathbb{R}^0 = \text{a point}$). Now if $f \neq 0$, write its power series as follows:

$$f(z_1, \ldots, z_n) = z_n^N g(z_1, \ldots, z_{n-1}) + z_n^{N+1} h(z_1, \ldots, z_n),$$

with $h$ and $g$ analytic and $g \neq 0$. Now divide out the $z_n^N$ and observe that

$$\frac{f}{z_n^N} \bigg|_{U \cap (\mathbb{R}^n \setminus \mathbb{R}^{n-1} \times \{0\})} = 0.$$  

So by continuity,

$$g + z_n h \big|_{U \cap (\mathbb{R}^{n-1} \times \{0\})} = 0$$

as well. In particular, since $z_n = 0$ on this part, we just get $g = 0$ on $U \cap (\mathbb{R}^{n-1} \times \{0\})$. By induction we conclude $g = 0$ everywhere, a contradiction. □

We also have a coordinate-free version of this lemma, namely:

**Lemma 2.** Let $V$ be a finite-dimensional $\mathbb{C}$-vector space and $W$ an $\mathbb{R}$-subspace with $V = W \oplus iW$. If $f : V \to \mathbb{C}$ is analytic and $f|_W = 0$, then $f = 0$.

We omit the proof; note that the decomposition $V = W \oplus iW$ ensures that an $\mathbb{R}$-basis for $W$ is also (grouping its elements together in pairs) a $\mathbb{C}$-basis for $V$. We apply this lemma to our study of $n \times n$ matrices in the following form:

**Lemma 3.** If $f : G \to \mathbb{C}$ is analytic and $f|_K = 0$, then $f = 0$.

**Proof.** Pull $f$ back to $\text{Mat}_{n \times n}(\mathbb{C})$, defining $g(X) = f(\exp(i \cdot X))$. Then $g : \text{Mat}_{n \times n}(\mathbb{C}) \to \mathbb{C}$ is analytic and $g = 0$ on the set of Hermitian matrices. Hence $g = 0$. □

We will use this trick multiple times: restricting or reducing to a compact subgroup in order to conclude something about the whole group (which isn’t compact). Here are some applications of the above:

**Application 4.** Say $V, W$ are analytic $G$-representations. Then $\mathcal{H}\text{om}_G(V, W) = \mathcal{H}\text{om}_K(V|_K, W|_K)$.

**Proof.** Given a linear map $A \in \mathcal{H}\text{om}(V, W)$, saying that $A$ commutes with the $G$-action, $A \in \mathcal{H}\text{om}_G(V, W)$, is just the statement $A \cdot \rho_V(g) = \rho_W(g) \cdot A$. This is an equality of analytic functions of $g$; so, by our lemma, equality holds on $G$ if and only if it holds on $K$. □
In other words, Hom-spaces don’t change under restricting to a compact subgroup.

**Application 5.** Say $V, W$ are analytic $G$-representations. Then

\[ V \cong W \text{ (as } G\text{-reps)} \iff V|_K \cong W|_K \text{ (as } K\text{-reps)}. \]

**Proof.** The left-hand statement is equivalent to the existence of a square matrix of full rank in $\text{Hom}_G(V, W)$. The right-hand statement is analogous, but with $\text{Hom}_K(V|_K, W|_K)$. So, apply the previous application. \hfill \Box

**Application 6.** Let $V$ be an analytic $G$-representation. If $W$ is a $K$-subrepresentation, then $W$ is also a $G$-subrepresentation.

**Proof.** Pick linear maps $\lambda_1, \ldots, \lambda_r : V \to \mathbb{C}$ such that $W = \bigcap \ker(\lambda_i)$. We want to show $\rho_V(g) \cdot w \in W$ for all $g \in G$. Thus, it’s sufficient to show that $\lambda_j(\rho_V(g) \cdot w) = 0$ for each $j$. Each of these is an analytic function of $g$. \hfill \Box

**Corollary 7.** Given $V$ an analytic $G$-representation, $V$ is simple (over $G$) if and only if $V|_K$ is simple over $K$.

**Corollary 8.** Every analytic $G$-representation is a direct sum of simple $G$-representations.

This last part is actually hard to do without passing to a compact group! In particular, it is hard to give a purely algebraic proof of this.

A high-level way of describing our results so far is to say that if you start with something from $G$, you can just study it on $K$. It’s not obvious that we can go the other way, i.e. that $K$-representations extend to $G$-representations. Our next goal, therefore, is to show that every continuous $K$-representation lifts to a rational $G$-representation.

### 2. Cautionary examples

Here’s a cautionary example of what can go wrong with other groups $G, K$: let $G$ instead be the (projective) elliptic curve $y^2 = x^3 - 1$. This is an abelian group, so all of its irreducible representations are one-dimensional. Let $K$ be its set of real points, so that $K \cong \mathbb{R}/\mathbb{Z} \cong S^1$. We know there is a representation of $\mathbb{R}/\mathbb{Z}$ given by $\theta \mapsto e^{2\pi i \theta}$. But, by compactness of $G$, we know there are no nonconstant analytic maps $G \to \mathbb{C}^*$!

The problem here is that although the functions will extend out from $K$, they don’t extend to all of $G$: they have branches that cause problems when you wrap around $G$. (One way to lift them successfully would be to go to a covering space of $G$).

**Paragraph added by David** Here is another example. The group $SL_3(\mathbb{R})$ injects into $PSL_3(\mathbb{C})$. (The kernel of $SL_3(\mathbb{C}) \to PSL_3(\mathbb{C})$ is the three element subgroup $\zeta^r \cdot \text{Id}_3$, where $\zeta$ is a third root of unity.) There are representations of $SL_3(\mathbb{R})$, for example the standard action on $\mathbb{C}^3$, which do not extend to $PSL_3(\mathbb{C})$, because they require functions which are only defined on $SL_3(\mathbb{C})$.

**Paragraph added by David** A deep theorem is that the above two examples cover all issues. Specifically, let $G$ be a connected complex Lie group, and $K$ a real subgroup with $T_e G = T_e K \oplus iT_e K$, where $T_e X$ is the tangent space to $X$ at the identity. If $G$ is affine (ruling out the elliptic curve example) and $K$ is compact (ruling out the $PSL_3$ example) then every continuous representation of $K$ extends to an analytic representation of $G$. David doesn’t know a conceptual proof of this theorem; only one that works by classifying all possible pairs $(G, K)$ and seeing that they work. We will not be proving this.

**Paragraph added by David, on material not mentioned in class** Given a compact real Lie group $K$, it turns out that $O(K)$ is a finitely generated $\mathbb{C}$ algebra, and Spec $O(K)$ is a complex Lie group $G$ with a natural embedding $K \hookrightarrow G$ for which the continuous representation theory of $K$ and the analytic representation theory of $G$ are equivalent. We will not be proving this either.

### 3. Lifting $K$-representations to $G$-representations

We go back to $G = GL_n, K = U(n)$. Our goal is to prove:

**Theorem 9.** Let $V$ be a continuous $K$-representation. Then $K$ lifts to a rational $G$-representation.
We begin by analyzing the characters of representations of the unitary group $K$. These will eventually give us a ‘hint’ as to how to find the appropriate rational representations of $G$. We first analyze $\chi_V$ on the compact torus $S$, bearing in mind that every unitary matrix is diagonalizable (and $\chi_V$ is a class function).

**Lemma 10.** Let $V$ be a continuous $K$-representation. Then $\chi_V : S \to \mathbb{C}$ is a symmetric Laurent polynomial in the eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_n}$.

**Proof.** We know $V|_S$ breaks up as a direct sum of $S$-simple representations. Since $S$ is abelian, every simple representation of it is one-dimensional: each is easily seen to be of the form

$$e^{i\theta_1}, \ldots, e^{i\theta_n} \mapsto e^{i(k_1\theta_1 + \cdots + k_n\theta_n)}$$

for some $k_1, \ldots, k_n \in \mathbb{Z}$. This shows $\chi_V|_S$ is a Laurent polynomial in the $e^{i\theta_j}$. To see that it is symmetric, consider a permutation $w \in S_n \subset U(n)$. Then we have

$$w \cdot \begin{pmatrix} e^{i\theta_1} & \cdots & e^{i\theta_n} \\ e^{i\theta_2} & \cdots & e^{i\theta_n} \\ \vdots & \ddots & \vdots \\ e^{i\theta_n} & \cdots & e^{i\theta_1} \end{pmatrix} w^{-1} = \begin{pmatrix} e^{i\theta_{w(1)}} & \cdots & e^{i\theta_{w(n)}} \\ \vdots & \ddots & \vdots \\ e^{i\theta_{w(n)}} & \cdots & e^{i\theta_{w(1)}} \end{pmatrix}$$

We know $\chi_V$ is a class function, so we conclude that

$$\chi_V(e^{i\theta_1}, \ldots, e^{i\theta_n}) = \chi_V(e^{i\theta_{w(1)}}, \ldots, e^{i\theta_{w(n)}}).$$

Thus $\chi_V|_S$ is symmetric. \qed

Now we begin the lifting process: we find some representations of $G$ whose characters are close to what we’re looking for. We now know to look for symmetric Laurent polynomials in the eigenvalues $x_1, \ldots, x_n$ of our matrices, so we’ll show something stronger: we can obtain any such function.

**Lemma 11.** For any $f \in \Lambda_n^\pm$, there are rational representations $W^+$ and $W^-$ of $G$ such that

$$\chi_{W^+}|_S - \chi_{W^-}|_S = f.$$

**Proof.** Clear denominators: we know that, for some $N$, $(x_1 \cdots x_n)^N f \in \Lambda_n$. Write this in the $e$ basis and separate the terms with positive and negative coefficients, as in

$$(x_1 \cdots x_n)^N f = \sum_{\lambda} c_{\lambda} e_\lambda - \sum_{\lambda} d_{\lambda} e_\lambda,$$

with $c_{\lambda}, d_{\lambda} \geq 0$. Note that if $\mathbb{C}^n$ is the obvious representation of $G$, then $\chi_{\mathbb{C}^n} = e_k(x_1, \ldots, x_n)$. As a special case, the character of the determinant representation is $x_1 \cdots x_n$. So, set

$$U^+ = \bigoplus_{\lambda} \left( \bigwedge_{1}^{\lambda_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge_{n}^{\lambda_n} \mathbb{C}^n \right)^{\otimes c_{\lambda}} , \quad W^+ = (\det)^{-N} \otimes U^+.$$

Observe that the character of $W^+$ is precisely $(x_1 \cdots x_n)^{-N} \sum_{\lambda} c_{\lambda} e_\lambda$. Similarly, set

$$U^- = \bigoplus_{\lambda} \left( \bigwedge_{1}^{\lambda_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge_{n}^{\lambda_n} \mathbb{C}^n \right)^{\otimes d_{\lambda}} , \quad W^- = (\det)^{-N} \otimes U^-.$$

Then $W^+$ and $W^-$ are the desired rational representations of $G$. \qed

Now that we’ve found the “right” characters and representations, our lifting theorem follows easily. We get to $K$ (by diagonalization), then to $G$ (by analyticity).

**Proof of Theorem 9.** By Lemma 4, the restriction of $\chi_V$ to $S$ is in $\Lambda_n^\pm$. So, by the previous lemma, we can find rational $G$-representations $W^+$ and $W^-$ such that

$$\chi_V|_S = \chi_{W^+}|_S - \chi_{W^-}|_S.$$

We know every unitary matrix is diagonalizable, so we can decompose $K$ as

$$K = \bigcup_{k \in K} kSk^{-1}.$$
This shows that, in fact, equality holds on all of $K$: $\chi_V|_K = \chi_{W^+}|_K - \chi_{W^-}|_K$. Since we know that representations of compact groups are determined by their characters, we have:

$$V \oplus W^- \cong W^+$$

as $K$-representations. In particular, $V$ is a $K$-subrep of $W^+$, hence also a $G$-subrep of $W^+$.

The same approach yields a similar lifting property for polynomial representations.

**Lemma 12.** If $f \in \Lambda_n$, then there exist polynomial $G$-representations $W^+$ and $W^-$ such that $\chi_{W^+} - \chi_{W^-} = f$. If $V$ is a $K$-representation such that $\chi_V$ is in $\Lambda_n$, then $V$ lifts to a polynomial representation of $G$.

Thus we make the following conclusion: characters of polynomial $\text{GL}_n$-irreducible representations span $\Lambda_n$, hence (by linear independence) are a basis for it. Similarly, characters of rational $\text{GL}_n$-irreducible representations are a basis for $\Lambda_n^\pm$. Our basis works in each degree separately. So we deduce a nice numerical consequence:

$$\#\{\text{poly irreps of } \text{GL}_n \text{ such that the character has degree } d\} = \#\{\text{partitions}(d)\}.$$