NOTES FOR OCTOBER 12
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The goal for today’s lecture is to prove:

**Theorem 1.** The characters of polynomial $GL_n$ irreps are the Schur functions.

The key will be to prove the following Peter-Weyl-like theorem

**Theorem 2.** Consider the polynomial ring in $n^2$ variables $z_{ij}$. As a $GL_n \times GL_n$ representation, we have

$$C[z_{ij}] \cong \bigoplus_{V \text{ a polynomial irrep}} V^\vee \otimes V.$$ 

As in Peter-Weyl, this sum means to take each isomorphism class once.

We continue the abbreviations

$$G = GL_n \quad K = U(n) \quad T = \{\text{diag}(z_1, \ldots, z_n) : z_i \in \mathbb{C}^*\} \quad S = K \cap T = \{\text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})\}.$$ 

1. **Proof of Theorem 2**

We have a map $C[z_{ij}] \to C^0(K)$ by restricting functions to the unitary group. Since polynomials in the $z_{ij}$ are analytic functions, this map is injective by the key lemma from last time. We claim that it lands in $O(K)$. Proof: $C[z_{ij}] = \bigoplus_d C[z_{ij}]_d$, where $C[z_{ij}]_d$ is homogeneous polynomials of degree $d$. Now, $C[z_{ij}]_d$ is clearly a finite dimensional $K \times K$ subrep of $C^0(K)$. So, by results from October 8, it is in $O(K)$.

Therefore, $C[z_{ij}] \cong \bigoplus_{V \in S} V^\vee \otimes V$ for some set $S$ of simple representations of $K$. We now must determine what the set $S$ is.

Let $V$ occur in $C[z_{ij}]$. Looking at the $1 \times G$ action on $V$, it is clear that $V$ is a polynomial $G$ rep. So every representation $V \in S$ is the restriction of a polynomial representation of $G$.

On the other hand, if $V$ is a polynomial representation of $G$, then the embedding $\text{End}(V)^\vee \to C^0(G)$ clearly lands in $C[z_{ij}]$. Explicitly, we are saying that $\chi(\rho_V(g))$ is a polynomial in the $z$’s, given that the entries of $\rho_V(g)$ are such a polynomial; that is obvious.

So we conclude that $S$ is the set of polynomial representations of $G$ as desired.

2. **A combinatorial consequence**

Consider both sides of Theorem 2 as $T \times T$ representations. To be precise, we are going to be acting by $\text{diag}(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}) \times \text{diag}(y_1, y_2, \ldots, y_n)$. (The inverses in the first term are precisely there to cancel the inverses defining the action of $G \times G$ on $C^0(G)$.)

On the left hand side, $z_{ij}$ transforms by $x_i y_j$. So the character of the left hand side is

$$\prod_{1 \leq i,j \leq n} \frac{1}{1 - x_i y_j}.$$ 

On the right hand side, $\text{diag}(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}) \times \text{diag}(y_1, y_2, \ldots, y_n)$ acts on $V^\vee \otimes V$ by

$$\chi_{V^\vee}(x_1^{-1}, \ldots, x_n^{-1})\chi_V(y_1, \ldots, y_n) = \chi_V(x_1, \ldots, x_n)\chi_V(y_1, \ldots, y_n).$$

So we deduce

$$\prod_{1 \leq i,j \leq n} \frac{1}{1 - x_i y_j} = \sum_{V \text{ a polynomial irrep}} \chi_V(x_1, \ldots, x_n)\chi_V(y_1, \ldots, y_n).$$

3. **Finishing the proof**

We would like to deduce that the $\chi_V$ are the Schur functions. There are two ways to finish the proof from here, both slightly more awkward than I would like.
Method 1. From a homework problem, \( \chi_V(x_1, \ldots, x_n) \) is a homogenous polynomial. As we noted in the previous class, we already know that the number of polynomial irreps of degree \( d \) is equal to the number of partitions of \( d \). By a lemma proved way back on September 12, this means that the \( \chi_V \) are self dual. Also, \( \chi_V \) is in \( \Lambda \) by the previous class. By another lemma from September 12, a self dual basis of \( \Lambda \) must be \( \pm s_\lambda \). It is clear that \( \chi_V \) has nonnegative coefficients, so the plus sign is correct. □

Method 2. We don’t really need to know that the number of degree \( d \) polynomial irreps is \( p(d) \). Indeed, if \( f_i \) is any family of symmetric polynomials with integer coefficients obeying \( \prod 1/(1 - x_i y_j) = \sum f_i(x) f_i(y) \), then I claim that the list of \( f_i \) contains each \( \pm s_\lambda \) exactly once, plus possibly some occurrences of the 0 function. Proof sketch: Let \( f_i = \sum a_i s_\lambda \). Comparing coefficients of \( s_\lambda(x) s_\lambda(y) \), we see that \( \sum_i a_i^2 = 1 \). So, for fixed \( \lambda \), exactly one \( a_{i,\lambda} \) is \( \pm 1 \) and the rest are zero. Comparing coefficients of \( s_\lambda(x) s_\mu(y) \), we see that, for fixed \( i \), at most one \( a_{i,\lambda} \) is nonzero. So the \( \chi_V \) are \( \pm \) the \( s_\lambda \), and maybe some zero functions. But it is clear that the \( \chi_V \) are nonzero and have nonnegative coefficients, so again we win. □

4. Concluding Comments

- If we look at the coordinate ring of \( GL_n \), namely \( \mathbb{C}[z_{ij}][\text{det}^{-1}] \), we get \( \bigoplus V^\vee \otimes V \) where the sum is over rational representations.
- The characters of the rational irreps are of the form

\[
(x_1 x_2 \ldots x_n)^{-N} s_\lambda(x_1, \ldots, x_n).
\]

Proof: Just tensor with a high power of the determinant representation to make it into a polynomial representation. We have

\[
s_{(\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1)}(x_1, x_2, \ldots, x_n) = (x_1 x_2 \ldots x_n) s_{\lambda_1, \lambda_2, \ldots, \lambda_n}(x_1, x_2, \ldots, x_n).
\]

As a result, the same symmetric Laurent polynomial can be expressed using more than one pair \((\lambda, N)\) as above. A nonredundant indexing set is the set of integer sequences \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \), where we do \textbf{not} impose that \( \mu_n \geq 0 \). The correspondence is that \( \mu_i = \lambda_i - N \).

- It follows immediately from the above that \( \langle \chi_V, \chi_W \rangle = \dim \text{Hom}_G(V, W) \), since the Schurs are orthonormal.
- We can look at \( \mathbb{C}[z_{ij}] \) where \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) as a \( GL_m \times GL_n \) rep. We have the equality of generating functions

\[
\prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x_1, \ldots, x_m) s_\lambda(y_1, \ldots, y_n).
\]

(Just take the identity in infinitely many variables and stick in 0 for the appropriate \( x \) and \( y \) variables.) So

\[
\mathbb{C}[z_{ij}] \cong \bigoplus_{\lambda} V_{\lambda}(m) \otimes V_{\lambda}(n)
\]

where \( V_{\lambda}(m) \) is the representation of \( GL_m \) with character \( s_\lambda(x_1, \ldots, x_m) \). The summands with \( \ell(\lambda) > \min(m,n) \) are zero, so we can equivalently write

\[
\mathbb{C}[z_{ij}] \cong \bigoplus_{\ell(\lambda) < \min(m,n)} V_{\lambda}(m) \otimes V_{\lambda}(n).
\]