NOTES FOR OCTOBER 17, 2012

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We spent some time discussing Problem 5d on Problem Set 5. The analogous finite group statement is the following:

Let \( V \) be a faithful \( G \)-representation. Set \( W = V \oplus V^\vee \oplus \mathbb{C} \). Then every irreducible representation \( U \) occurs in \( W^\otimes N \) for \( N \gg 0 \).

1. Some comments from last time

\[ \mathbb{C}[x_{ij}] = \bigoplus V^\vee \otimes V, \]

where \( V \) varies over polynomial irreducible representations of \( \text{GL}_n \) and the equality above is taken as \( (\text{GL}_n \times \text{GL}_n) \)-representations. We want to point out that we similarly have:

\[ \mathbb{C}[z_{ij}] = 1 \leq i \leq m, \ 1 \leq j < n = \bigoplus_{\lambda, \ell(\lambda) \leq \min(m,n)} V^\vee_\lambda (m) \otimes V_\lambda (n) \]

as \( (\text{GL}_n \times \text{GL}_n) \)-representations.

Proof. We have

\[ \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{\lambda} S_\lambda(x_1, \ldots, x_m) S_\lambda(y_1, \ldots, y_n). \]

Note that \( S_\lambda(x_1, \ldots, x_m) = 0 \) if \( \ell(\lambda) > m \) and \( S_\lambda(y_1, \ldots, y_n) = 0 \) if \( \ell(\lambda) > n \). The result follows. \( \square \)

Last time that we showed that we have a bijective correspondence between polynomial \( \text{GL}_n \) representations and partitions \( \lambda \) with \( \ell(\lambda) \leq n \); and the character of \( V_\lambda \) is \( s_\lambda \). We want to point out that this implies:

\[ \dim \text{Hom}_{\text{GL}_n}(V, W) = \langle \chi_V, \chi_W \rangle. \]

(Consider \( \chi_V \) and \( \chi_W \) as polynomials in \( \Lambda \) using the combinatorially defined inner product on \( \Lambda \).) Note that if \( |\lambda| > n \), the we use the standard inclusion \( \Lambda_n \hookrightarrow \Lambda \).

Notation. \( V_\lambda \) or \( V_\lambda (n) \) is the \( \text{GL}_n \)-representation with character \( S_\lambda(x_1, \ldots, x_n) \).

2. How we will construct \( V_\lambda \)

Recall that

\[ h_\lambda = s_\lambda + \sum_{\mu \prec \lambda} \kappa_{\lambda \mu} s_\mu, \]

\[ e_\lambda^T = s_\lambda + \sum_{\mu \succ \lambda} \kappa_{\lambda T \mu}^T s_\mu. \]

So the equality

\[ \langle h_\lambda, e_\lambda^T \rangle = 1 \]

comes from the \( s_\lambda \) term.

Let

\[ H = \bigotimes_k \text{Sym}_{\lambda_k}^n V, \]

\[ E = \bigotimes_k \Lambda_{\lambda_k^T}^n V. \]
So $\chi_H = h_\lambda$ and $\chi_E = e^{\lambda^T}$. We see that

$$H = V_\lambda \oplus \bigoplus_{\mu < \lambda} V^{\oplus K_\lambda}_\mu,$$

$$E = V_\lambda \oplus \bigoplus_{\mu > \lambda} V^{\oplus K_\lambda^T}_\mu.$$

So $\text{Hom}_{\text{GL}(V)}(E, H) \cong \mathbb{C}$ and if $\varphi$ is a $\text{GL}(V)$-equivariant homomorphism $E \to H$, then $\text{Im}(\varphi) \cong V_\lambda$. Our next goal will be to describe such a map $\varphi$ explicitly.