Problem 1 Use the mathematical software package of your choice to do these computations:
(a) Write down the 6 transition matrices relating the $e$, $h$ and $m$ bases for the degree 4 symmetric polynomials. (These should be $5 \times 5$ matrices.)
(b) Expand $(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4)$ in the $e$-basis of $\Lambda_4$.

Problem 2 Let $\lambda$ and $\mu$ be two partitions with $|\lambda| = |\mu|$. Show that $\lambda \succeq \mu$ if and only if $\mu^T \succeq \lambda^T$.

Problem 3 Let $\phi : \text{GL}_2(\mathbb{C}) \to \mathbb{C}$ be a continuous function which obeys $\phi(ghg^{-1}) = \phi(h)$. Show that $\phi \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) = \phi \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$.

Problem 5 (a) Show that $\prod_{i,j} (1 + x_i y_j) = \sum_\lambda e_\lambda(x)m_\lambda(y)$.
(b) Define $b_{\lambda\mu} = \langle e_\lambda, h_\mu \rangle$. Show that $b_{\lambda\mu} = b_{\mu\lambda}$.
(c) Show that the involution $\omega$ preserves $\langle \ , \ \rangle$.
(d) In class, we gave an interpretation of $\langle h_\lambda, h_\mu \rangle$ in terms of matrices of nonnegative integers with given row and column sum. Give a similar interpretation of $b_{\lambda\mu}$.

Problem 6 This problem will work you through the basic properties of the power symmetric functions; they are important but which won’t come up very often for us. It will provide our first (but not best) proof that $\langle \ , \ \rangle$ is positive definite.
Define $p_k(x) = \sum x_i^k$ and define $p_\lambda(x) = \prod_i p_{\lambda_i}(x)$.
(a) Prove Newton’s Identity:
$$ke_k = e_{k-1}p_1 - e_{k-2}p_2 + e_{k-3}p_3 - \cdots \pm e_1 p_{k-1} \mp p_k.$$  
(b) Show that $\Lambda \otimes \mathbb{Q}$ is $\mathbb{Q}[p_1, p_2, p_3, \ldots]$.
(c) Establish a formula of the form
$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda z_\lambda p_\lambda(x)p_\lambda(y)$$
for some positive rational numbers $z_\lambda$. Hint: $\frac{1}{1-w} = \exp(w + w^2/2 + w^3/3 + w^4/4 + \cdots)$.
(d) What is $\langle p_\lambda, p_\mu \rangle$?