Problem Set 10: Due Friday, December 8

See the course website for homework policy. This is the last problem set!

1. Let $W$ be a finite Coxeter group and let $V$, $R$, $S$, $S_+$ and $\Delta$ have their usual meanings. Let $A = R/RS_+$ be the coinvariant algebra.

   (a) We define the socle of $A$ to be $\{ a \in A : va = 0 \ \forall v \in V \}$. Show that the socle of $A$ is one dimensional, spanned by $\Delta$. (Hint: Let $a$ lie in the socle, and consider $\partial_{\delta}(\beta a)$.)

   (b) For any integer $i$ between 0 and $\ell(w_0)$, multiplication defines a bilinear map $A_i \times A_{\ell(w_0)−i} \to A_{\ell(w_0)} \cong \mathbb{R}$. Show that this is a perfect pairing.

2. Let $s_1$, $s_2$, $\ldots$, $s_{n−1}$ be the standard Coxeter generators of $S_n$. As we will discuss in class, a Coxeter element is a permutation $c$ with a reduced word of the form $c = s_{i_1}s_{i_2}\cdots s_{i_{n−1}}$, where $(i_1, i_2, \ldots, i_{n−1})$ is a permutation of $(1, 2, \ldots, n−1)$.

   (a) Show that $c$ is an $n$-cycle $(a_1a_2\cdots a_n)$ and describe how to read off $(a_1a_2\cdots a_n)$ from $(i_1, i_2, \ldots, i_{n−1})$.

   (b) Let $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{C}^n$ be a nonzero $e^{2\pi i/n}$ eigenvector of $c$. Describe the geometry of the $n$ points $v_1, v_2, \ldots, v_n \in \mathbb{C}$.

   (c) We focus on the particular case $c = s_1s_3s_5\cdots s_2s_4s_6\cdots$. Let $H \subset \mathbb{R}^n$ be the 2-plane of vectors of the form $(\Re(\alpha v_1), \Re(\alpha v_2), \ldots, \Re(\alpha v_n))$ for $\alpha \in \mathbb{C}$. Describe the intersection of $H$ with the fundamental domain $D = \{(x_1, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n \}.$

3. Let $\Gamma$ be a graph with $n$ vertices. An orientation $O$ of $\Gamma$ is an assignment of a direction $i \to j$ to each edge $(i, j)$ of $\Gamma$. A sink of $(\Gamma, O)$ is a vertex $i$ where all adjacent edges are directed into $i$, and a sink is a vertex where all adjacent vertices are directed out. A sink-source reversal takes a sink and reverses all edges incident to it, making a source. Given a cycle $\gamma = (v_1, v_2, \ldots, v_k)$ in $\Gamma$ and an orientation $O$ of $\Gamma$, the flow of $O$ along $\Gamma$ is

   $$\text{Flow}(\gamma, O) = \#\{i : v_i \to v_{i+1}\} − \#\{i : v_i \leftarrow v_{i+1}\}$$

where the indices are cyclic modulo $k$. An orientation $O$ is called acyclic if $−k < \text{Flow}(\gamma, O) < k$ for every length $k$ cycle $\gamma$ in $\Gamma$.

The aim of this problem is to prove the following result: If $O_1$ and $O_2$ are two acyclic orientation of $\Gamma$, and $\text{Flow}(\gamma, O_1) = \text{Flow}(\gamma, O_2)$ for all cycles $\gamma$, then we can transform $O_1$ to $O_2$ by a sequence of sink-source reversals. We write $i \to j$ for the orientation $O_1$ and similarly for $O_2$. With all that as prelude, we begin:

   (a) Prove the converse: If $O_1$ can be transformed into $O_2$ by a sequence of sink-source reversals, then $\text{Flow}(\gamma, O_1) = \text{Flow}(\gamma, O_2)$ for all cycles $\gamma$.

   We may, and do, reduce to the case that $\Gamma$ is connected.

   (b) Let $O_1$ and $O_2$ be two acyclic orientation of $\Gamma$. Suppose that $\text{Flow}(\gamma, O_1) = \text{Flow}(\gamma, O_2)$ for all cycles $\gamma$. Show that there is a unique function $h$ from the vertices of $\Gamma$ to $\mathbb{Z}_{\geq 0}$ such that
1. For every edge \((i, j)\) of \(\Gamma\),

\[
    h(i) - h(j) = \begin{cases} 
        1 & \text{if } i \leftarrow 1 j \text{ and } i \rightarrow 2 j \\
        0 & \text{if } (i \rightarrow 1 j \text{ and } i \rightarrow 2 j) \text{ or } (i \leftarrow 1 j \text{ and } i \leftarrow 2 j) \\
        -1 & \text{if } i \rightarrow 1 j \text{ and } i \leftarrow 2 j
    \end{cases}
\]

2. For every vertex \(v\) of \(\Gamma\), we have \(h(v) \geq 0\) and, for at least one vertex, we have \(h(v) = 0\).

We define \(d(\mathcal{O}_1, \mathcal{O}_2) = \sum_{v \in \Gamma} h(v)\). Our proof is by induction on \(d\).

(c) Explain why the base case, \(d(\mathcal{O}_1, \mathcal{O}_2) = 0\), holds.

(d) Let \(d > 0\) and set \(H = \max_v h(v)\). Show that there is some \(v_0\) with \(h(v) = H\) which is a sink of \(\mathcal{O}_1\).

(e) Let \(\mathcal{O}_1'\) be the orientation obtained from \(\mathcal{O}_1\) by a sink-source reversal at \(v_0\). Show that

\[
    d(\mathcal{O}_1', \mathcal{O}_2) = d(\mathcal{O}_1, \mathcal{O}_2) - 1.
\]