See the course website for homework policy.

1. Let $V$ be a finite dimensional real vector space equipped with a positive definite symmetric bilinear form $\cdot$. Let $\Lambda$ be a discrete additive subgroup of $V$ with $\text{Span}_R(\Lambda) = V$. Define $G$ to be the group of linear transformations $g : V \to V$ with $g(u) \cdot g(v) = u \cdot v$ and $g(\Lambda) = \Lambda$.
   
   (a) Show that $G$ is finite.
   (b) Show that, for $g \in G$, we have $\text{Tr} \, g \in \mathbb{Z}$.
   (c) Let $\dim V = 2$ and let $g \in G$. Show that $g$ has order $1$, $2$, $3$, $4$ or $6$.

2. This problem will explore a representation where the $\alpha_i$ and $\alpha_i^\vee$ are not linearly independent. $V$ and $V^\vee$ be $3$ dimensional, written as column and row vectors respectfully, and take

\[
\begin{align*}
\alpha_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & \alpha_2 &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, & \alpha_3 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & \alpha_4 &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \\
\alpha_1^\vee &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}, & \alpha_2^\vee &= \begin{bmatrix} -2 & 0 & 0 \end{bmatrix}, & \alpha_3^\vee &= \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}, & \alpha_4^\vee &= \begin{bmatrix} 0 & -2 & 0 \end{bmatrix}.
\end{align*}
\]

We define $D = \{ x \in V^\vee : \langle x, \alpha_i \rangle \geq 0 \}$. Recall that $s_i$ acts on $V^\vee$ by

\[
s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i^\vee.
\]

(a) Show that the matrix $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is a Cartan matrix. What are the $m_{ij}$?

(b) Let $V_1^\vee$ be the affine linear space $\{ \begin{bmatrix} x \\ y \end{bmatrix} \}$ in $V_1^\vee$. Show that $W$ preserves $V_1^\vee$.

(c) In terms of the coordinates $(x, y)$ on $V_1^\vee$, write down the action of the $s_i$ on $V_1^\vee$. Give inequalities on $x$ and $y$ describing $D_1 := D \cap V_1^\vee$.

(d) Draw and label the domains $wD_1$ in the two dimensional plane $V_1^\vee$ for several values of $w$.

3. This problem describes a different representation of $\tilde{A}_{n-1}$ from the one on Problem Set 2.

Let $n \geq 3$ be a positive integer. Let $V$ be the vector space of sequences $(a_i)_{i \in \mathbb{Z}}$ such that

\[
a_{i+n} - a_i \text{ is a constant independent of } i.
\]

Let $\tilde{A}_{n-1}$ act on $V$ by $w(a)_i = a_{w^{-1}(i)}$.

(a) Choose a basis for $V$, and write the matrices of $s_1$, $s_2$, $\ldots$, $s_n$ in your basis.

(b) Give explicit vectors $\alpha_i \in V$ and $\alpha_i^\vee \in V^\vee$ such that $s_i(x) = x - \langle \alpha_i^\vee, \alpha_i \rangle \alpha_i$. Choose your signs such that $\langle \alpha_i^\vee, \alpha_i \rangle$ is positive on the point $x_i = i$.

(c) Compute the Cartan matrix $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.

Once again, let $D = \{ x \in V^\vee : \langle x, \alpha_i \rangle \geq 0, \ 1 \leq i \leq n \}$.

Let $\bar{V}$ be the quotient of $V$ by the vector space of constant sequences. Let $\bar{V}_1$ be the affine subspace $a_{i+n} = a_i + 1$ of $\bar{V}$. Note that $\dim \bar{V}_1 = n - 1$, which means we can draw it for $n = 3$. I’ll write $\bar{D}$ for the image of $D$ in $\bar{V}$ and $\bar{D}_1$ for the intersection $\bar{D} \cap \bar{V}_1$.

(d) For $n = 3$, draw $\bar{D}_1$, $s_1\bar{D}_1$, $s_2\bar{D}_1$, $s_3\bar{D}_1$, $s_3s_2s_1\bar{D}_1$, $s_1s_2s_1\bar{D}_1$ inside $\bar{V}_1$. Draw the hyperplanes fixed by $s_1$, $s_2$, $s_3$.

(e) Show that $\bar{V}_1 = \bigcup_{w \in \tilde{A}_{3-1}} w\bar{D}_1$. 