**Problem Set Five: Due Friday, October 23**

**Problem 1.** Let $M$ be a matrix. We define a top justified consecutive minor of $M$ to be a minor of the form
\[ \Delta_{(a+1)(a+2)\ldots(a+k)}(M), \]
for some $a$ and $k$, and a left justified consecutive minor to be a minor of the form
\[ \Delta_{12\ldots k}(M) \]
for some $b$ and $k$. We’ll define a top-or-left justified minor to be a minor which is either top or left justified (or both).

(1) Show that a totally positive matrix is determined by the values of its top-or-left justified minors.

Let $X$ be a $n \times n$ totally positive matrix. Let $D$ be the diagonal matrix with entries
\[ \frac{\Delta_1^1(X)}{\Delta_1^1(X)}, \frac{\Delta_{12}^{12}(X)}{\Delta_{12}^{12}(X)}, \ldots, \frac{\Delta_{[n-1]}^{[n-1]}(X)}{\Delta_{[n-1]}^{[n-1]}(X)}. \]
Let $L$ be the unique lower triangular matrix such that
\[ \Delta_{12\ldots k}^{(b+1)(b+2)\ldots(b+k)}(L) = \frac{\Delta_{12\ldots k}^{(b+1)(b+2)\ldots(b+k)}(X)}{\Delta_{12\ldots k}^{12\ldots k}(X)} \]
and let $U$ be the unique upper triangular matrix such that
\[ \Delta_{12\ldots k}^{(a+1)(a+2)\ldots(a+k)}(U) = \frac{\Delta_{12\ldots k}^{12\ldots k}(X)}{\Delta_{12\ldots k}^{(a+1)(a+2)\ldots(a+k)}(X)}. \]

(2) Show that $X = LDU$.

We have now confirmed the claim from earlier in the course that, if $X$ is totally positive and has $LDU$ factorization $(L, D, U)$, then $L, D$ and $U$ are totally nonnegative.

**Problem 2.** The Plücker coordinates on the Grassmannian $G(2, 4)$ obey the relation $\Delta^{13}\Delta^{24} = \Delta^{12}\Delta^{34} + \Delta^{14}\Delta^{23}$. A point of $G(2, 4)$ is called totally nonnegative if all of its Plücker coordinates are nonnegative.

(1) Which of the subsets of $\{12, 13, 14, 23, 24, 34\}$ are capable of being the sets $\{I : \Delta^I(x) \neq 0\}$ for $x$ a totally nonnegative point of $G(2, 4)$? You should find that there are 33 in total.

(2) Give examples of points of $G(2, 4)$ where each of the following occurs
\[ \Delta^{12}, \Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24}, \Delta^{34} > 0 \]
\[ \Delta^{12}, \Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24} > 0, \quad \Delta^{34} = 0 \]
\[ \Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24} > 0, \quad \Delta^{12} = \Delta^{34} = 0 \]
\[ \Delta^{12}, \Delta^{13}, \Delta^{14} > 0 \quad \Delta^{23} = \Delta^{24} = \Delta^{34} = 0 \]
\[ \Delta^{12}, \Delta^{13}, \Delta^{23} > 0 \quad \Delta^{14} = \Delta^{24} = \Delta^{34} = 0 \]