Problem Set 5: Due Friday, October 11

See the course website for homework policy.

1. Let $G$ be the group of signed $n \times n$ permutation matrices for $n \geq 3$. I number the generators of $G$ as $s_1, s_2, \ldots, s_n$ so that $m_{12} = 4$ and $m_{23} = m_{34} = \cdots = m_{(n-1)n} = 3$. Let $G$ be $G_n \ltimes \mathbb{Z}^n$ with $G_n$ acting on $\mathbb{Z}^n$ by these matrices; we consider $G$ as a group of Euclidean symmetries of $\mathbb{R}^n$. In this problem, we will play with subgroups of $G$ to get experience with Euclidean symmetry groups.

(a) Show that $G$ is generated by affine reflections. Is it $\tilde{C}_n$ or $\tilde{B}_n$?
(b) Let $G_1$ be the subgroup $D_n \ltimes \mathbb{Z}^n$ of $G_n \ltimes \mathbb{Z}^n$. Show that $G_1$ is not generated by affine reflections.
(c) Let $G_2$ be the subgroup of $G_1$ generated by affine reflections. Show that $G_2$ is $D_n \ltimes \Lambda$ for a sublattice $\Lambda$ of $\mathbb{Z}^n$ and describe $\Lambda$ explicitly.
(d) Let $\chi : C_n \to \{\pm 1\}$ be the unique character with $\chi(s_1) = 1$ and $\chi(s_2) = \chi(s_3) = \cdots = \chi(s_n) = -1$. Let $G_3$ be the subset of $G$ consisting of pairs $(w, (a_1, \ldots, a_n))$ where $\chi(w) = (-1)\sum a_i$. Show that $G_3$ is a subgroup of $G$.
(e) Construct a short exact sequence $1 \to \Lambda \to G_3 \to C_n \to 1$ where $\Lambda$ is a free rank $\mathbb{Z}$-module with a $C_n$ action, and show that this sequence is not semidirect. (Hint: Find an order 2 element in $C_n$ that doesn’t lift to an order 2 element of $G_3$.)

2. Let $W$ be a finite crystallographic Coxeter group with a chosen crystallographic root system of rank $n$. Let $\tilde{W}$ be the affine Coxeter group $W \ltimes \mathbb{Z}(\alpha_i^\vee)$. Recall that the affine hyperplane arrangement is the arrangement of hyperplanes $\{x \in V^\vee : \langle x, \beta \rangle = k\}$ for $\beta \in \Phi$ and $k \in \mathbb{Z}$.

Let $D$ be the fundamental domain in this hyperplane arrangement.

(a) Show that every region of the hyperplane arrangement is of the form $wD + \gamma^\vee$ for precisely one $w \in W$ and $\gamma^\vee \in \mathbb{Z}(\alpha_i^\vee)$.
(b) Let $\Pi$ be the parallelepiped $\{x \in V^\vee : 0 \leq \langle x, \alpha_i \rangle \leq 1\}$ for $1 \leq i \leq n$. Let $M$ be the number of regions into which the affine hyperplane arrangement divides $\Pi$. Show that the index $[\mathbb{Z}(\alpha_i^\vee) : \mathbb{Z}(\alpha_i^\vee)]$ is equal to $\frac{|W|}{M}$.
(c) The domain $D$ is a simplex with walls $\{x \in V^\vee : \langle x, \alpha_i \rangle = 0\}$ for $1 \leq i \leq n$ and one additional wall $\{x \in V^\vee : \langle x, \theta \rangle = 1\}$ for some root $\theta$. Let $\theta = \sum c_i \alpha_i$. Show that $M = \frac{\text{Vol}(\Pi)}{\text{Vol}(D)} = n!c_1c_2 \cdots c_n$.
(d) Multiplying the formulas above gives $|W| = n!c_1 \cdots c_n[\mathbb{Z}(\alpha_i^\vee) : \mathbb{Z}(\alpha_i^\vee)]$. Check this formula in type $\tilde{A}_n$.

3. For a Coxeter group $W$, put $W(q) = \sum_{w \in W} q^{\ell(w)}$. The combinatorics of this generating function is a fascinating topic we won’t have time for, so we glimpse it here.

In this problem, we derive a recursion for $W(q)$ in terms of parabolic subgroups. We recall the notation $W_I$ for the parabolic subgroup and $I^W = \{w \in W : s_i$ is a left ascent of $w$ for all $i \in I\}$ from Problem Set 4. We write $s_1, s_2, \ldots, s_n$ for the simple generators of $W$ and $[n] = \{1, 2, \ldots, n\}$.

(a) Show that

$$W_I(q) \cdot \left( \sum_{w \in I^W} q^{\ell(w)} \right) = W(q)$$

(b) Show that

$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \sum_{w \in I^W} q^{\ell(w)} = \begin{cases} q^{\ell(w_0)} & W \text{ finite} \\ 0 & W \text{ infinite} \end{cases}.$$

(c) Combine the two parts above to give a recursion for $W(q)$ in terms of $W_I(q)$ for $I \subseteq [n]$, and show that $W(q)$ is always a rational function of $q$.

(d) Use the above formula to compute $\sum_{w \in \tilde{A}_2} q^{\ell(w)}$. You may take as known that $\sum_{w \in A_2} q^{\ell(w)} = 1 + 2q + 2q^2 + q^3$. 