Problem Set 6: Due Friday, October 27

See the course website for homework policy.

Note: Because I expect the course to be providing an overview of Lie groups at this point, the problems are not closely tied to the current material. They are either things that I think it would be nice for you to know, or things which will be useful when we start studying invariant theory.

1. Let $\Phi$ be a finite root system with simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$.
   (a) Let $s_i$ be a simple generator and $\beta$ a positive root other than $\alpha_i$. Show that $s_i\beta$ is a positive root. (Hint: Quote an earlier problem.)
   Define $\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$.
   (b) Show that $\langle \alpha_i^\vee, \rho \rangle = 1$ for $1 \leq i \leq n$.
   (c) Take our usual models of the finite classical crystallographic root systems: \{\pm $e_i - e_j$\} in type A, \{\pm $e_i \pm e_j$, $e_k$\} in type B, \{\pm $e_i \pm e_j$, $2e_k$\} in type C and \{\pm $e_i \pm e_j$\} in type D and compute $\rho$ in these cases.
   For a reflection $t \in T$, let $\beta_t$ be the positive root with $t\beta_t = -\beta_t$.
   (d) For any $w \in W$, show that $w(\rho) = \rho - \sum_{t \in \text{inv}(w)} \beta_t$.

2. This problem describes a model for the group $\tilde{C}_n$ similar to our model for $\tilde{A}_{n-1}$ in terms of affine permutations. We define $\tilde{C}_n$ by its Coxeter diagram:

   \[
   \begin{array}{cccccccc}
   s_0 & 4 & s_1 & \cdots & s_{n-1} & 4 & s_n \\
   \end{array}
   \]

   Let $Z = \{k + 1/2 : k \in \mathbb{Z}\}$. Let $\tilde{S}^C_n$ be the set of bijections $w : Z \to Z$ such that

   \[w(z + 2n) = w(z) + 2n \text{ and } w(-z) = -w(z).\]

   Our goal will be to show $\tilde{S}^C_n \cong \tilde{C}_n$.
   (a) Describe a map $\tilde{C}_n \to \tilde{S}^C_n$ by describing where to send the generators $s_0, s_1, \ldots, s_n$.
   Let $V^\vee$ be the vector space of functions $a : Z \to \mathbb{R}$ obeying $a(-z) = -a(z)$ and such that $a(z + 2n) - a(z)$ is a constant independent of $z$.
   (b) Choose a basis for $V^\vee$ and write down the action of $s_0, \ldots, s_n$ in that basis. Write $s_i(x)$ in the form $x - \langle \alpha_i, x \rangle \alpha_i^\vee$ for some roots $\alpha_i$ and $\alpha_i^\vee$.
   (c) Check that the $\alpha_i$ and $\alpha_i^\vee$ pair by the Cartan matrix for $\tilde{C}_n$.
   (d) Show that $\tilde{C}_n \to \tilde{S}^C_n$ is an isomorphism.
3. Let $W$ be a Coxeter group with generators $s_1, s_2, \ldots, s_n$. For $I \subseteq [n]$, we defined $W_I$ to be the subgroup generated by $\{s_i : i \in I\}$. We define $^I W$ and $W^I$ by

$$^I W = \{ w \in W : s_i \text{ is a left ascent of } w \text{ for all } i \in I \}$$

$$W^I = \{ w \in W : s_i \text{ is a right ascent of } w \text{ for all } i \in I \}.$$  

(a) Show that each $w \in W$ can be factored as $uv$, with $u \in W_I$ and $v \in ^I W$.

(b) Show that we have $v \in ^I W$ if and only if $\text{inv}(v) \cap W_I = \emptyset$.

(c) Suppose that $w = uv$ with $u \in W_I$ and $v \in ^I W$. Show that $\text{inv}(u) = \text{inv}(w) \cap W_I$.

(d) Let $w \in W$. Show that there is a unique factorization of $w$ as $w = uv$ with $u \in W_I$ and $v \in ^I W$. The unique $u$ and $v$ are denoted $w_I$ and $^I w$. We define $^I w$ and $w^I$ similarly.

(e) The next two pages of this problem set feature the hyperplane arrangement $A_3$ (drawn in stereographic projection) with regions $D$, $s_1 D$, $s_2 D$, $s_3 D$ labeled. Set $I = \{1, 3\}$.

On the first page, use four colors to indicate which $w$'s have $w_I$ equal to 1, $s_1$, $s_3$ or $s_1 s_3$. On the second page, outline the cosets $uW_I$ and color according to the value of $^I w$. 
