Problem 1. We return to the topic of the strong order on $S_n$. Given a permutation matrix $w$, let $r_{ab}(w)$ be the rank of the upper left $a \times b$ submatrix of $w$. We define $u \leq v$ in strong order iff $r_{ab}(u) \geq r_{ab}(v)$ for all $(a, b)$.

We showed before that, for any permutation $w$ and any transposition $(i j)$, we have $w(i j) > w$ if $w(i) < w(j)$ and $w(i j) < w$ if $w(i) > w(j)$. We want to show that, if $u < w$, then there is a transposition $(i j)$ with $u < u(i j) < w$. Thus, any two elements of $S_n$ are linked by a chain of transpositions in strong order.

Let $i$ be the least index for which $u(i) \neq w(i)$.

(1) Show that $u(i) < w(i)$.

Let $j$ be the least index for which $u(i) < u(j) \leq w(i)$.

(2) Show that $u < u(i j) \leq w$.

Problem 2. Show that $u \leq w$, in strong order, if and only if $B_- u B_+ \cap B_+ w B_+$ is nonempty.

The next problem uses the notion of a Kasteleyn labelling, which we will discuss in class. Here is the definition for those who want to start earlier: Let $G$ be a bipartite planar graph embedded in the plane, and all of whose bounded faces are discs. A Kasteleyn labelling of $G$ is an assignment of a nonzero complex number $\alpha(e)$ to each edge $e$ of $G$ with the following property: Consider any face of $G$, with edges $e_1, e_2, \ldots, e_{2k}$. Then we impose that

$$\alpha(e_1)\alpha(e_3)\alpha(e_5)\cdots\alpha(e_{2k-1}) = (-1)^{k-1}\alpha(e_2)\alpha(e_4)\alpha(e_6)\cdots\alpha(e_{2k}).$$

Problem 3. There is a cute geometric way to find Kasteleyn labellings. Embed $G$ in the complex plane such that, for every bounded face $F$ of $G$, the vertices of $F$ lie on a circle. For an edge $e$ with white endpoint $w$ and black endpoint $b$, let $\alpha(e)$ be the unit complex number in direction $w - b$. Prove that this is a Kasteleyn labelling.