Problem Set 7: Due Friday, November 1

See the course website for homework policy.

1. Read the course notes from October 18 to November 30. Suggest something that can be improved, or a question they raise.

2. This question fills in the details of how permutahedra work. Take a Coxeter group and Cartan matrix as usual, and take roots and coroots that pair by the Cartan matrix such that both \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) and \((\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee)\) are linearly independent. Choose \(\rho \in V\) such that \(\langle \rho, \alpha_i^\vee \rangle > 0\) for all \(i\) (since the \(\alpha_i^\vee\) are linearly independent, this is possible). Let \(X = \{w\rho\}_{w \in W}\) and let \(P\) be the convex hull of \(X\). The polytope \(P\) is called the \(W\)-permutahedron.

   (a) Show that \(\rho - w\rho\) is a positive linear combination of \(\{\beta_t : t \in \text{inv}(w)\}\). Hint: Induction on \(\ell(w)\).

   (b) Let \(\theta \in V^\vee\), so \(\langle \theta, \cdot \rangle\) is a linear function on \(P\). Show that this function achieves its minimum at \(\rho\), and nowhere else, if and only if \(\theta \in D^\circ\). (Note that \(D^\circ\) is nonempty since the \(\alpha_i\) are linearly independent.)

   (c) Show that \(\langle \theta, \cdot \rangle\) achieves its minimum at \(w\rho\), and nowhere else, if and only if \(\theta \in wD^\circ\).

   (d) Let \(W_I\) be a standard finite parabolic subgroup of \(W\) and let \(D^\circ_I\) be the corresponding face of \(D\). Let \(P_I\) be the convex hull of \(\{w\rho : w \in W_I\}\). Show that \(\langle \theta, \cdot \rangle\) achieves its minimum on \(P_I\) and nowhere else if and only if \(\theta \in D^\circ_I\). Analogously, show that \(\langle \theta, \cdot \rangle\) achieves its maximum on \(uP_I\) and nowhere else if and only if \(\theta \in uD^\circ_I\).

   (e) Take the standard root system for \(B_3\), with simple roots \(e_1, e_2 - e_1\) and \(e_3 - e_2\). Sketch the three dimensional polytope \(P\).

   (f) The previous example is for a finite Coxeter group, which is the usual context in which permutahedra are considered. But we can handle infinite groups. Consider the Cartan matrix \(\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\) with roots \(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\) and \(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\) and co-roots \(\begin{bmatrix} 1 & -1 \end{bmatrix}\) and \(\begin{bmatrix} -1 & 1 \end{bmatrix}\). Take \(\rho = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\). Show that \(X\) is the set of vectors of the form \(\begin{bmatrix} -(k-1)k \\ -(k+1)k \end{bmatrix}\) for \(k \in \mathbb{Z}\), and sketch the infinite “polytope” \(P\).

The final question is on the back.
3. This question fills in the details of Loday’s construction. Let $T_n$ be the set of planar binary trees with leaves $\ell_0, \ell_1, \ldots, \ell_n$. We draw the leaves at the bottom of the tree from left to right.

We number the internal vertices of the tree as $v_1, v_2, \ldots, v_n$ from left to right, so that $v_k$ is to the right of $\ell_0, \ell_1, \ldots, \ell_{k-1}$ and to the left of $\ell_k, \ell_{k+1}, \ldots, \ell_n$.

![Diagram of a binary tree]

Given a tree $T$, define the integer vector $c(T)$ where $c(T)_i$ is the number of ordered pairs $(j,k)$ such that $\ell_j$ is a left descendant of $v_i$ and $\ell_k$ is a right descendant of $v_i$. In the example above, $c(T) = (2, 1, 3)$. Let $\text{Assoc}_n$ be the convex hull of the vectors $c(T)$ for $T \in T_n$.

Given $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with the $x_i$ distinct, let $T(\vec{x})$ be the unique binary tree which can be drawn such that $v_i$ is at height $x_i$. We will first be showing that $\langle \vec{x}, \cdot \rangle$ is maximized on $\text{Assoc}_n$ at the vertex $c(T(\vec{x}))$.

(a) Show that there is a continuous function $\phi: \mathbb{R}^n \to \mathbb{R}$ such that, if $\vec{x}$ is a vector with distinct entries, we have $\phi(\vec{x}) = \langle \vec{x}, c(T(\vec{x})) \rangle$.

(b) Let $\vec{x}$ and $\vec{y}$ be two vectors in $\mathbb{R}^n$ so that, for any $t \in \mathbb{R}$, the vector $t\vec{x} + (1 - t)\vec{y}$ has at most two equal entries. Show that the restriction of $\phi$ to the line segment from $\vec{x}$ to $\vec{y}$ is convex. (Hint: This is a piecewise linear function; describe what happens at its corners.)

(c) Let $\vec{x}$ be a vector in $\mathbb{R}^n$ with distinct entries and let $U$ be a tree other than $T(\vec{x})$. Show that $\langle \vec{x}, c(T(\vec{y})) \rangle > \langle \vec{x}, c(U) \rangle$. Hint: Choose a generic $\vec{y}$ such that $U = T(\vec{y})$ and consider what happens to $\phi$ on the line segment from $\vec{x}$ to $\vec{y}$.

(d) Let $\vec{x}$ be a vector in $\mathbb{R}^n$. Show that $\langle \vec{x}, \cdot \rangle$, on $\text{Assoc}_n$, is maximized solely at $c(T(\vec{x}))$. This in particular shows that every $c(T)$ is a vertex of $\text{Assoc}_n$.

We now know that the normal fan to $\text{Assoc}_n$ is given by coarsening the $S_n$ hyperplane arrangement as described in class.

(e) Let $T$ be a binary tree and let $\text{Cone}(T)$ be the cone of all $\vec{x}$ with $T(\vec{x}) = T$. Show that $\text{Cone}(T)$ and $\text{Cone}(U)$ border along a codimension 1 face if and only if $c(T)$ and $c(U)$ differ by a single association.