See the course website for homework policy.

1. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we adopt $x^a$ as shorthand for $\prod x_i^{a_i}$.
   (a) Let the Coxeter group $B_n$ act on $\mathbb{R}^n$ in the standard manner, and recall that we computed
   that $\rho = (1/2, 3/2, \ldots, n - 1/2)$ in this case. Show that
   \[
   \sum_{w \in B_n} (-1)^{\ell(w)} x^w(\rho) = \det \left[ \begin{array}{cccc}
   x_1^{1/2} - x_1^{-1/2} & x_2^{1/2} - x_2^{-1/2} & \cdots & x_n^{1/2} - x_n^{-1/2} \\
   x_1^{3/2} - x_1^{-3/2} & x_2^{3/2} - x_2^{-3/2} & \cdots & x_n^{3/2} - x_n^{-3/2} \\
   \vdots & \vdots & \ddots & \vdots \\
   x_1^{n-1/2} - x_1^{-n+1/2} & x_2^{n-1/2} - x_2^{-n+1/2} & \cdots & x_n^{n-1/2} - x_n^{-n+1/2}
   \end{array} \right]
   \]
   (b) Let the Coxeter group $C_n$ act on $\mathbb{R}^n$ in the standard manner, and recall that we computed
   that $\rho = (1, 2, \ldots, n)$ in this case. Show that
   \[
   \sum_{w \in C_n} (-1)^{\ell(w)} x^w(\rho) = \det \left[ \begin{array}{cccc}
   x_1^{1} - x_1^{-1} & x_2^{1} - x_2^{-1} & \cdots & x_n^{1} - x_n^{-1} \\
   x_1^{2} - x_1^{-2} & x_2^{2} - x_2^{-2} & \cdots & x_n^{2} - x_n^{-2} \\
   \vdots & \vdots & \ddots & \vdots \\
   x_1^{n} - x_1^{-n} & x_2^{n} - x_2^{-n} & \cdots & x_n^{n} - x_n^{-n}
   \end{array} \right]
   \]
   (c) Let the Coxeter group $D_n$ act on $\mathbb{R}^n$ in the standard manner, and recall that we computed
   that $\rho = (0, 1, 2, \ldots, n - 1)$ in this case. Show that
   \[
   \sum_{w \in D_n} (-1)^{\ell(w)} x^w(\rho) = \det \left[ \begin{array}{cccc}
   1 & 1 & \cdots & 1 \\
   x_1^{1} + x_1^{-1} & x_2^{1} + x_2^{-1} & \cdots & x_n^{1} + x_n^{-1} \\
   x_1^{2} + x_1^{-2} & x_2^{2} + x_2^{-2} & \cdots & x_n^{2} + x_n^{-2} \\
   \vdots & \vdots & \ddots & \vdots \\
   x_1^{n} + x_1^{-n} & x_2^{n} + x_2^{-n} & \cdots & x_n^{n} + x_n^{-n}
   \end{array} \right]
   \]

2. Let $W$ be a Coxeter group and $W_I$ a parabolic subgroup. We recall the minimal coset representatives $W^I$ from Problem Set 6, Problem 3. We put $[n] = \{1, 2, 3, \ldots, n\}$.
   (a) Show that
   \[
   \sum_{w \in W} q^{\ell(w)} = \left( \sum_{w \in W_I} q^{\ell(w)} \right) \left( \sum_{w \in W^I} q^{\ell(w)} \right).
   \]
   Take $W = S_n$ and $W_I = S_k \times S_{n-k}$ (in the obvious way).
   (b) Give a bijection between $W^I$ and the $k$-element subsets of $[n]$ such that, if $w \in W^I$
   corresponds to $A \subset [n]$, then $\ell(w) = \# \{(a, b) \in A \times ([n] \setminus A) : a > b\}$.
   (c) Explain what formula you have proved for
   \[
   \sum_{A \subset [n]} q^{\# \{(a, b) \in A \times ([n] \setminus A) : a > b\}}.
   \]