1. Conclusion of the Proof of the Ratio of Alternants Formula

Last time, we defined $\rho = (n-1, n-2, \cdots, 2, 1, 0)$. We have an action of $S_n$ on $\mathbb{Z}^n$ by

$$v^*(\alpha) = v(\alpha) + v(\rho) - \rho$$

To finish the proof of the ratio of alternants formula, we must show the following.

**Lemma 1.**

$$\sum_{v \in S_n} (-1)^v K_{\lambda v^*}(\alpha) = \begin{cases} 1 & \lambda = \alpha \\ 0 & \text{otherwise} \end{cases}$$

We will prove this by a sign-cancelling involution. Let $T_{high}$ be the SSYT of shape $\lambda$ with all 1’s in its first row, all 2’s in its second row, and so on.

$$T_{high} = \begin{array}{cccc} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots \\ n \end{array}$$

Then $T_{high}$ contributes to the $\alpha = \lambda$ term. Our involution will be defined on $SSYT(\lambda) \setminus \{T_{high}\}$, and will switch each tableau of content $\gamma$ with one that has content $s_i^*(\gamma)$ for some $s_i$.

Recall that $s_i$ is the permutation that switches $i$ and $i+1$, so the action of $s_i^*$ corresponds to keeping one $i+1$, and switching the rest of the $i$’s and $i+1$’s. Thus

$$s_i^*(\gamma_1, \cdots, \gamma_i, \gamma_{i+1}, \cdots, \gamma_n) = (\gamma_1, \cdots, \gamma_{i+1}-1, \gamma_i+1, \cdots, \gamma_n)$$

For example, if $(\gamma_i, \gamma_{i+1})$ is $(2,10)$, $s_i^*(\gamma_i, \gamma_{i+1})$ will be $(9,3)$.

Now look at a tableau $T$ of shape $\Lambda$, and find the highest row that doesn’t match $T_{high}$. Look at the last element of that row (shown in blue). This will be $i+1$. Let $T$ contain $r$ copies of $i$ and $s$ copies of $i+1$.

The boxes labeled $i$ and $i+1$ in $T$ form a smaller tableau within $T$ (shown in red below), containing $r$ copies of $i$ and $s-1$ copies of $i+1$. We switch this inner tableau with one that has $s-1$ copies of $i$’s and $r$ copies of $i+1$ using the Bender-Knuth involution. So the new tableau as a whole contains $s-1$ copies of $i$ and $r+1$ copies of $i+1$. 

\[\begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array}\]

\[\begin{array}{cc} 1 & 2 \\ 3 & 2 \end{array}\]

\[\begin{array}{cc} 1 & 3 \\ 2 & 3 \end{array}\]

\[\begin{array}{cc} 1 & 3 \\ 3 & 3 \end{array}\]
This process changes the content of $T$ by $s_i^*$. There are no $i$’s or $i+1$’s in the rows above the row where we differed from $T_{\text{high}}$. If the difference occurs in the $k^{\text{th}}$ row, then $i+1 > k$. The elements in previous rows still match $T_{\text{high}}$. Elements in previous rows are 1, 2, · · · , $k-1$. So the highest rows still match $T_{\text{high}}$. The $k^{\text{th}}$ row still doesn’t match $T_i$ because it still has the rightmost element, so the operation is self-inverting. The lemma is proved, and the the ratio of alternants formula follows.

2. Schur functions are orthonormal

We recall the Jacobi-Trudi formula

$$s_\mu = \begin{vmatrix}
  h_{\mu_1} & h_{\mu_1+1} & \cdots & h_{\mu_1+(n-1)} \\
  h_{\mu_2-1} & h_{\mu_2} & \cdots & \vdots \\
  \vdots & \vdots & \ddots & h_{\mu_n-(n-1)} \\
  h_{\mu_n-(n-1)} & & & h_{\mu_n}
\end{vmatrix} = \sum_{w \in S_n} (-1)^w h_w^*(\mu)$$

(Note: for $\alpha \in \mathbb{Z}_{\geq 0}^n$, the notation $h_\alpha$ means $h_{\text{sort} (\alpha)}$. For example, $s_{21} = \begin{vmatrix}
  h_2 & h_3 \\
  h_0 & h_1
\end{vmatrix} = h_{21} - h_{03}$)

We define

$$L_{\lambda\mu} = \sum_{w \in S_n} \begin{cases} (-1)^w & \text{if } w^*(\mu) \text{ is a permutation of } \lambda \\ 0 & \text{else} \end{cases}$$

so we have

$$s_\mu = \sum_{\lambda \text{ a partition}} L_{\lambda\mu} h_\lambda.$$

We also just proved

$$\sum_{w \in S_n} (-1)^w K_{\lambda w^*(\alpha)} = \begin{cases} 1 & \lambda = 0 \\ 0 & \text{else} \end{cases}$$

so

$$\sum_{\nu} K_{\lambda\nu} L_{\nu\alpha} = \delta_{\lambda\alpha}.$$ 

If two square matrices obey $AB = \text{Id}$, they also\footnote{It often happens in combinatorics that $AB = \text{Id}$ has a simple combinatorial proof and $BA = \text{Id}$ does not. Loehr and Mendes, Bijective matrix algebra, Linear Algebra Appl. 416 (2006), no. 2-3, 917–944 give an algorithm which, given as input a sign canceling involution proving $AB = \text{Id}$, produces as output a sign canceling involution proving $BA = \text{Id}$. Loehr and Mendes motivating example was to find a combinatorial interpretation of $\sum_k L_{\lambda\kappa} K_{\kappa\mu} = \delta_{\lambda\mu}$. To my knowledge, no simpler interpretation is known. See also the discussion at http://sbseminar.wordpress.com/2010/07/26/a-proof-length-challenge/} obey $BA = \text{Id}$, so

$$\sum_{\kappa} L_{\lambda\kappa} K_{\kappa\mu} = \delta_{\lambda\mu}.$$ 

As a consequence

$$\sum_{\kappa} L_{\lambda\kappa} s_\kappa = m_\lambda.$$ 

Let $\langle , \rangle$ be our standard inner product. Let $( , )$ be the inner product where the Schur functions are orthonormal. Then

$$\langle m_{\lambda}, s_\mu \rangle = \left( m_{\lambda}, \sum_{\lambda'} L_{\lambda\mu} h_{\lambda'} \right) = L_{\lambda\mu} \quad \text{and} \quad (m_{\lambda}, s_\mu) = \left( \sum_{\kappa} L_{\lambda\kappa} s_\kappa, s_\mu \right) = L_{\lambda\mu}$$

By linearity, we have $\langle f, g \rangle = (f, g)$ for all $f, g \in \Lambda$. □
3. Concluding remarks

This paragraph added by David. I can’t remember whether I made this argument on Sept 21 or Sept 14, but it should be recorded somewhere. Let $L$ be the part of $\Lambda$ in degree $n$ for some fixed $n$, so $L \cong \mathbb{Z}^N$ as an abelian group. Suppose that we have some other orthonormal basis $t_\lambda$ for $L$. Then $t_\lambda$ is some permutation of $\pm s_\lambda$. Proof: Write $t_\lambda = \sum c_{\lambda\mu} s_\mu$. Then $\langle t_\lambda, t_\lambda \rangle = \sum_{\mu} c_{\lambda\mu}^2 = 1$, indicating that precisely one of the $c_{\lambda\mu}$ is $\pm 1$ and all others are zero. So $t_\lambda = \lambda s_\lambda'$ for some $\lambda'$.

Moreover, if $\lambda \neq \mu$, then $\langle t_\lambda, t_\mu \rangle = 0$, so $\lambda' \neq \mu'$ and we see that $\lambda \mapsto \lambda'$ is a permutation.

Something David should have mentioned but didn’t. Suppose that we have a sequence of homogeneous symmetric polynomials $t_i$ with integer coefficients, obeying $\prod (1 - x_i y_j)^{-1} = \sum t_i(x) t_i(y)$. Then the same proof shows $t_i$ contains $\pm s_\lambda$ once for each $\lambda$, plus possibly some copies of the zero polynomial. We don’t need to check first that $t_i$ is a basis for $\Lambda$.

On problem set 1, problem 5, we show that $\omega$ preserves the inner product on $\Lambda$. So $\omega(s_\lambda) = \pm s_{\lambda'}$ for some $\lambda'$. On problem set 3, problem 3, we will show that $\omega(s_\lambda) = s_\lambda^T$. Sketch of proof: The Jacobi-Trudi identity gives $s_\lambda$ as a determinant in the $h$’s, so we can express $\omega(s_\lambda)$ as a determinant in the $e$’s. Problem 3 evaluates that determinant and shows that it is $s_{\lambda'}$.

3.1. A remark just for the fun of it. Suppose we have $L \cong \mathbb{Z}^N$, with inner product $L \times L \to \mathbb{Z}$ which is symmetric, positive definite, and has dual bases $e_i$ and $f_i$ both in $L$. We might expect that this would force a self-dual basis in $L$, but this is not the case.

For a counterexample, consider the matrix

$$
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 2
\end{bmatrix}
$$

corresponding to the graph

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The matrix is symmetric, positive definite, and has determinant 1. But if $\langle \ , \ \rangle$ is the inner product corresponding to this matrix, then for any $v = (v_1, \ldots, v_8)$ we have

$$
\langle v, v \rangle = 2 \sum v_i^2 - 2 \sum_{(i,j) \text{ in graph}} v_i v_j
$$

and is hence even for all $v$ (so in particular cannot be equal to 1).