Let $\lambda, \mu$ be partitions such that $\lambda \supset \mu$, that is $\lambda_i \geq \mu_i$ for all $i$. The skew Young diagram $\lambda/\mu$ is the set difference between the two partitions.

**Example 1.** Let $\lambda = 6322$ and $\mu = 411$.

![Young diagrams](image)

Then $\lambda/\mu$ is a skew Young diagram.

A (skew) semi-standard Young tableau (SSYT) of shape $\lambda/\mu$ is a filling of the Young diagram $\lambda/\mu$ with positive integers such that the rows are weakly increasing and the columns are strictly increasing.

**Example 2.** A semi-standard Young tableau of shape $\lambda/\mu$.

![Young tableau](image)

The skew Schur function $s_{\lambda/\mu}$ is defined

$$s_{\lambda/\mu} = \sum_{SSYT \, T \, \text{shape} (T) = \lambda/\mu} x^T.$$ 

**Proposition 1.** Skew Schur functions are symmetric.

**Proof.** Same as that for Schur functions (via the Bender-Knuth involution). □

We now examine skew Schur functions with respect to various bases of $\Lambda$.

In the monomial basis, the skew Schur function is

$$s_{\lambda/\mu} = \sum_{\nu} K_{\lambda/\mu, \nu} m_{\nu}$$

where the coefficient $K_{\lambda/\mu, \nu}$ is the (skew) Kostka number. It equals the number of SSYT of shape $\lambda/\mu$ and content $\nu$.

Since ordinary Schur’s already span $\Lambda$, the skew Schur’s are linear combinations of these. The coefficients of this linear combination, as will be discussed below, are called Littlewood-Richardson numbers.

For the homogeneous basis, we have an analogue of the Jacobi-Trudi identity.

**Proposition 2** (Jacobi-Trudi).

$$s_{\lambda/\mu} = \det (h_{\lambda_i - \mu_j - i + j})$$

**Proof.** Same as that for Schur functions (non-intersecting lattice paths). □
Example 3.

\[ s_{32/1} = \begin{array}{|c|c|} \hline & \vrule \end{array} \]
\[ s_{32/1} = \begin{vmatrix} h_2 & h_4 \\ h_0 & h_2 \end{vmatrix} = h_{22} - h_4 \]

Also, we have \( \omega(s_{\lambda/\mu}) = s_{\lambda^T/\mu^T} \) and dual Jacobi-Trudi.

Any product of skew Schur functions is also a skew Schur function, as we can put two skew shapes together (disconnected-ly) to make a new skew shape for the product. In particular, any complete homogeneous polynomial \( h_{\lambda} \) is a skew Schur function. This is because

\[ h_{\lambda} = s_{(\lambda)} \]

where \( (\lambda) \) is a row of \( \lambda \) boxes. Then (for example) \( h_{422} = s_{(4)}s_{(2)}s_{(2)} \) is the skew Schur function for the shape

\[ \begin{array}{|c|c|c|} \hline & & \\
& & \\
& & \\
\end{array} \]

which has disconnected rows of length \( \lambda_1, \ldots, \lambda_\ell(\lambda) \).

There is also a relation between skew Schur and non-skew Schur functions.

**Proposition 3.** The “skew by \( \mu \)” operator that sends \( s_{\lambda} \mapsto s_{\lambda/\mu} \) is adjoint to multiplication by \( s_{\mu} \).

In other words, for any \( f \in \Lambda \)

\[ \langle s_{\lambda/\mu}, f \rangle = \langle s_{\lambda}, f s_{\mu} \rangle. \]

In particular, for a Schur function \( s_{\nu} \)

\[ \langle s_{\lambda/\mu}, s_{\nu} \rangle = \langle s_{\lambda}, s_{\mu}s_{\nu} \rangle. \]

Then the decomposition of \( s_{\lambda/\mu} \) into a sum of non-skew Schur functions \( s_{\nu} \) has structure constants that are the constants that you get from multiplying non-skew Schur functions.

Example 4. For the skew shape 321/21

we can compute \( s_{321/21} \). Because \( s_{321/21} \) is the product of the skew Schurs of its disconnected components

\[ s_{321/21} = (s_1)^3 = (s_2 + s_1) s_1 = (s_3 + s_21) = (s_21 + s_{111}) = s_3 + 2s_{21} + s_{111} \]

We can check that the coefficient of \( s_3 \) in \( s_{321/21} \) (expanded in the \( s \)-basis) is

\[ \langle s_{321/21}, s_3 \rangle = \langle s_{321}, s_3 s_{21} \rangle = 1 \]

because

\[ s_3 s_{21} = s_{321} + s_{411} + s_{42} + s_{51} \]

which has \( s_{321} \) with coefficient 1.

Recall the Pieri rule, which we use for some computations in the above. A horizontal \( k \)-strip is a skew shape with \( k \) boxes and no more than one box in each column. The Pieri rule is

\[ s_{\mu} h_k = \sum_{\lambda} s_{\lambda} \]

where the sum ranges over all partitions \( \lambda \) such that \( \lambda/\mu \) is a horizontal \( k \)-strip.

**Proof.** (of Proposition 3, adjointness) It suffices to prove this for the \( f \) from some basis for \( \Lambda \), so we will show it for \( f = h_{\nu} \). We wish to show that \( \langle s_{\lambda/\mu}, h_{\nu} \rangle = \langle s_{\lambda}, s_{\mu} h_{\nu} \rangle \).

On the LHS, we have

\[ \langle s_{\lambda/\mu}, h_{\nu} \rangle = \text{coefficient of} \ m_{\nu} \text{ if } s_{\lambda/\mu} \text{ is written in the } m \text{-basis} \]

\[ = K_{\lambda/\mu, \nu} \] (that is, the number of SSYT of shape \( \lambda/\mu \), content \( \nu \)).
On the RHS
\[ \langle s_\lambda, s_\mu h_\nu \rangle = \text{coefficient of } s_\lambda \text{ in } s_\mu h_\nu. \]

Use the Pieri rule to turn \( s_\mu h_\nu \) into a sum.

\[
s_\mu h_\nu = s_\mu h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k}
\]
\[
= \sum_{\rho(1)} s_{\rho(1)} h_{\nu_2} \cdots h_{\nu_k} \quad \left\{ \begin{array}{l}
\rho(1) \text{ s.t. } \rho(1)/\mu \text{ is a horizontal } \nu_1\text{-strip}
\end{array} \right.
\]
\[
= \sum_{\rho(1),\rho(2)} s_{\rho(2)} h_{\nu_3} \cdots h_{\nu_k} \quad \left\{ \begin{array}{l}
\rho(1) \text{ s.t. } \rho(1)/\mu \text{ is a horizontal } \nu_1\text{-strip}
\rho(2) \text{ s.t. } \rho(2)/\rho(1) \text{ is a horizontal } \nu_2\text{-strip}
\end{array} \right.
\]
\[
= \sum_{\rho(1),\rho(2),\ldots,\rho(k)} s_{\rho(k)} \quad \left\{ \begin{array}{l}
\rho(1) \text{ s.t. } \rho(1)/\mu \text{ is a horizontal } \nu_1\text{-strip}
\vdots
\rho(k) \text{ s.t. } \rho(k)/\rho(k-1) \text{ is a horizontal } \nu_k\text{-strip}
\end{array} \right.
\]

Each term \( s_{\rho(k)} \) in the sum represents a SSYT with shape \( \rho(k)/\mu \) and content \( \nu \); in the SSYT, each cell of the \( i \)th horizontal strip contains \( i \). That is,

\[
s_\mu h_\nu = \sum_{\text{SSYT}} s_{\rho(\text{shape}(T)=\rho/\mu,\text{content}(T)=\nu)}
\]

The coefficient of \( s_\lambda \) in \( s_\mu h_\nu \) is the number of SSYT with shape \( \lambda/\mu \) and content \( \nu \), i.e. \( K_{\lambda/\mu,\nu} \), and

\[
\langle s_\lambda, h_\nu \rangle = \langle s_\lambda, s_\mu h_\nu \rangle
\]
as desired. \( \square \)

**Example 5.** If \( \mu = (k) \), then

\[
\langle s_{\lambda/(k)}, s_\nu \rangle = \langle s_\lambda, s_\nu s_{(k)} \rangle = \left\{ \begin{array}{ll}
1 & \text{if } \lambda/\nu \text{ is a horizontal } k\text{-strip} \\
0 & \text{otherwise}
\end{array} \right.
\]

and

\[
s_{\lambda/(k)} = \sum_{\lambda/\nu \text{ is a horizontal } k\text{-strip}} s_\nu.
\]

With some actual numbers,

\[
53/2 = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

\[
s_{53/2} = s_{33} + s_{42} + s_{51}.
\]

The coefficients \( \langle s_\lambda/\mu, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle = c_{\lambda/\mu}^\nu \) are called **Littlewood-Richardson coefficients**. One can also define \( s_{\lambda/\mu} \) with the following relation.

\[
s_\lambda(x, y) = \sum_{\mu \subset \lambda} s_\mu(x) s_{\lambda/\mu}(y)
\]

This is basically the same as the definition by summing over skew SSYT. Take a SSYT of shape \( \lambda \) and consider the positions of the entries indexing \( x \) variables. These form some SSYT of some shape \( \mu \); the remaining \( y \)-variables form an SSYT of shape \( \lambda/\mu \).

We reprove the adjointness relation from this perspective.
Proof.

\[
\prod_{i,j}(1 - x_i z_j)^{-1} \times \prod_{i,j}(1 - y_i z_j)^{-1} = \sum_\lambda s_\lambda(x, y)s_\lambda(z) \\
= \sum_{\lambda, \mu} s_\mu(x)s_{\lambda/\mu}(y)s_\lambda(z) \\
= \sum_{\lambda, \mu, \nu} s_\mu(x)s_\nu(y)s_\lambda(z) \langle s_{\lambda/\mu}, s_\nu \rangle
\]

and

\[
\prod_{i,j}(1 - x_i z_j)^{-1} \times \prod_{i,j}(1 - y_i z_j)^{-1} = \left(\sum_\mu s_\mu(x)s_\mu(z)\right) \left(\sum_\nu s_\nu(y)s_\nu(z)\right) \\
= \sum_{\mu, \nu} s_\mu(x)s_\nu(y)s_\mu(z)s_\nu(z) \\
= \sum_{\lambda, \mu, \nu} s_\mu(x)s_\nu(y)s_\lambda(z) \langle s_\lambda, s_\mu s_\nu \rangle
\]

so \( \langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle \). □