1. The answer to the coffee problem

We want to compute

$$\prod_{k=2}^{\infty} \frac{k^3-1}{k^3}.$$

Set $\omega = e^{2\pi i/3}$. So we are computing

$$\prod_{k=2}^{\infty} \frac{(k-\omega)(k-\omega^2)(k-1)}{k^3}.$$

Taking the product out to n terms and canceling the $k-1$ in the numerator against k in the denominator, we want to compute

$$\lim_{n\to\infty} \frac{1}{n} \prod_{k=2}^{n} \frac{(k-\omega)(k-\omega^2)}{k^2}.$$

It turns out to be convenient to shift our indexing by 1 in the $k-\omega^2$ terms, producing

$$\lim_{n\to\infty} \frac{n-\omega^2}{n(1-\omega^2)} \prod_{k=2}^{n} \frac{(k-\omega)(k-1-\omega^2)}{k^2} = \frac{1}{1-\omega^2} \lim_{n\to\infty} \prod_{k=2}^{n} \frac{(k-\omega)(k-1-\omega^2)}{k^2}.$$

But now the key observation: $1+\omega^2 = -\omega$. So this is

$$\frac{1}{1-\omega} \lim_{n\to\infty} \prod_{k=2}^{n} \frac{(k-\omega)(k+\omega)}{k^2} = \frac{1}{(1-\omega^2)(1-\omega)(1+\omega)} \prod_{k=1}^{\infty} \left(1 - \frac{\omega^2}{k^2}\right).$$

We now use the Euler product identity,

$$\frac{\sin(\pi z)}{\pi z} = \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2}\right).$$

So our infinite product is

$$\frac{1}{(1-\omega^2)(1-\omega)(1+\omega)} \frac{\sin(\pi\omega)}{\pi\omega}.$$

Since $\omega = -1/2 + i\sqrt{3}/2$, we have $\sin(\pi\omega) = \sin(-\pi/2 + i\pi\sqrt{3}/2) = -\cos(i\pi\sqrt{3}/2) = -\cosh(\pi\sqrt{3}/2)$. And a quick computation gives that $(1-\omega^2)(1-\omega)(1+\omega)\omega = -3$. So our final answer is

$$\frac{\cosh(\pi\sqrt{3}/2)}{3\pi}.$$

2. A general observation

Another useful identity of Euler's is

$$\lim_{n\to\infty} n^{-z} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) = \frac{1}{\Gamma(1+z)}.$$

So, suppose you want to compute an infinite product of the form

$$\prod_{k=1}^{\infty} f(1/k)$$

where f is some polynomial. In order for the product to converge, f must be of the form $f(z) = 1 + f_2 z^2 + f_3 z^3 + \cdots$. Write $f(z) = (1 + \alpha_1 z)(1 + \alpha_2 z) \cdots (1 + \alpha_r z)$. The vanishing of the z terms means that we must have $\sum \alpha_i = 0$. So we can write our product as

$$\lim_{n \to \infty} n^{-\sum \alpha_i} \prod_{k=1}^{n} (1 + \alpha_1/k)(1 + \alpha_2/k) \cdots (1 + \alpha_r/k),$$

the exponent of n just being a fancy way of writing 0, and use Euler's identity to get

$$\frac{1}{\Gamma(1 + \alpha_1)\Gamma(1 + \alpha_2) \cdots \Gamma(1 + \alpha_r).}$$

In general, that's as far as you can get, but you should be on the lookout for opportunities to use the identities $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1-z)\Gamma(1+z) = \frac{\pi z}{\sin(\pi z)}$.

I started out along this route, but I had to deal with the nuisance that the product started at 2, not 1, and that $\Gamma(0) = \infty$, so I was dividing by ∞ at one point. Then I decided that there must be a way to get straight to the sin function without going through the Γ function, and I found the above.