Categorification of perfect matchings

Alastair King, Bath

work in progress with I. Canakci & M. Pressland [CKP]
and with B.T. Jensen & X. Su [JKS3]

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Toy model: perfect matchings on a circle

On a circular graph \((C_0, C_1)\) with \(n\) vertices, a \textbf{(perfect) matching} is a choice of orientation for each edge.

A matching is specified by its label \(J = (J_\bullet, J_\circ)\) with \(J_\bullet \subseteq C_1\) being the anti-clockwise edges and \(J_\circ\) the clockwise edges.

A matching has \textit{chirality} \(k = (k_\bullet, k_\circ) \in \mathbb{N}\{\bullet, \circ\}\), where \(k_\bullet = \#J_\bullet\) and \(k_\circ = \#J_\circ\), so that \(k_\bullet + k_\circ = n\).

A new matching of the same chirality is obtained by flipping a source to a sink or vice versa. Chirality is the only invariant of flipping.
Cochains on a closed string

Fatten the circle to a quiver with faces $Q = (Q_0, Q_1, Q_2)$, i.e. a 2-complex s.t. $\partial f$ is an oriented cycle, for all $f \in Q_2$.

For any such $Q$, a matching is a function $\mu \in \mathbb{N}^{Q_1}$ s.t. $d\mu = 1$, on all faces $f \in Q_2$, in ptic, in lattice $\mathbb{M} = \{\mu \in \mathbb{Z}^{Q_1} : d\mu \in c(\mathbb{Z})\}$.

$\mathbb{Z} \xrightarrow{c} \mathbb{Z}^{Q_0} \xrightarrow{d} \mathbb{Z}^{Q_1} \xrightarrow{d} \mathbb{Z}^{Q_2}$

$\mathbb{Z} \xrightarrow{c} \mathbb{Z}^{Q_0} \xrightarrow{d} \mathbb{M} \xrightarrow{\text{deg}} \mathbb{Z}$

$d$ is coboundary, $c$ is constants, $i$ is inclusion, $\text{deg}$ is restriction of $d$. Now flip is adding/subtracting $d(s_i)$ for $s_i$ basic in $\mathbb{Z}^{Q_0}$.

Denote by $\mathbb{M}^+ = \mathbb{M} \cap \mathbb{N}^{Q_1}$ the cone of multi-matchings.

For string, rank $\mathbb{M} = n + 1$ and $\mathbb{M}^+$ is the cone on a unit $n$-cube.
Another example: double (or $r$-fold) dimers

$\Sigma$

$\deg \mu = 2$

**Question:** why is $\text{wt}(\Sigma) = 2$ ? **Guess:** because $\chi(\mathbb{P}^1) = 2$ and some appropriate quiver Grassmannian is $\mathbb{P}^1$. 
Chirality revisited

Let $\mathbb{H}^1 = M/d(\mathbb{Z}Q_0)$ and $h: M \to \mathbb{H}^1$ be the quotient.

Note: $\text{deg} = \text{dg} \circ h$ for $\text{dg}: \mathbb{H}^1 \to \mathbb{Z}$ and $\text{dg}^{-1}(0) = H^1(Q)$.

For the closed string: $\mathbb{H}^1 \cong \{(h_\bullet, h_\circ) \in \mathbb{Z}\{\bullet, \circ\} : h_\bullet + h_\circ \in n\mathbb{Z}\}$.

Explicitly, write $Q_1 = Q_1^\bullet \cup Q_1^\circ$ and, for $\ast \in \{\bullet, \circ\}$, define two closed 1-cycles $a_\ast = \sum_{a \in Q_1^\ast} a$. Then $h_\ast(\mu) = \mu(a_\ast) = \sum_{a \in Q_1^\ast} \mu(a)$ and $\text{deg}(\mu) = \frac{1}{n}(h_\bullet(\mu) + h_\circ(\mu))$.

Fixed chirality $k = (k_\bullet, k_\circ)$ and define $M_k = h^{-1}\langle k \rangle$,
then rank $M_k = n$ and $\mathbb{Z} \xrightarrow{c} \mathbb{Z}Q_0 \xrightarrow{d} M_k \xrightarrow{\text{deg}} \mathbb{Z}$ is exact.

Fact: $M_k \cong$ the sublattices of the weight lattice of $GL(n)$ that grade the homogeneous coordinate rings $\mathbb{C}[\hat{\text{Gr}}^n_{k_\bullet}]$ and $\mathbb{C}[\hat{\text{Gr}}^n_{k_\circ}]$ and $\text{deg}$ gives the usual degree (Plücker coords $\Delta_J$ have $\text{deg} 1$).
Categorification of matchings

Let $Z = \mathbb{C}[[t]]$ and $Q = (Q_0, Q_1, Q_2)$ be a quiver with faces s.t.
(i) the associated topological space $|Q|$ is connected
(ii) every arrow $a \in Q_1$ is in the boundary of some face $f \in Q_2$
(iii) $Q$ admits virtual matchings, i.e. $\deg M \to \mathbb{Z}$ is surjective.

The category of matrix factorizations $MF(Q; t)$ consists of representations $M, \phi$ of $Q$ s.t.
(i) each $M_i : i \in Q_0$ is a f.g. free $Z$-module of rank $r := \text{rk } M$
(ii) the maps $\phi_a : M_{ta} \to M_{ha}$ satisfy $\phi_{a_s} \circ \cdots \circ \phi_{a_1} = t$,
    when $a_s + \cdots + a_1 = \partial f$ is the boundary of a face $f \in Q_2$.

There is a natural invariant $\nu : K(MF(Q; t)) \to \mathbb{M} : [M] \mapsto \nu_M$
given by $\nu_M(a) = \dim \text{coker } \phi_a$ and such that $\deg \nu_M = \text{rk } M$.

For each (deg 1) matching $\mu \in \mathbb{M}^+$, there is a rk 1 rep’n $M(\mu), \phi$
with $\nu_M(\mu) = \mu$, given by $M(\mu)_i = Z$ and $\phi_a = t^{\mu(a)}$.

A flip corresponds to simple extension/shortening of the rep’n.
The closed string categorified

For $x_j \in Q_1^\circ$, $y_j \in Q_1^\bullet$ and fixed $k = (k_\bullet, k_\circ)$, define $\mathbb{Z}$-algebra $C_k$ as path algebra $\mathbb{Z}Q$ mod $xy = t = yx$, $x^{k_\bullet} = y^{k_\circ}$ ($\Rightarrow x^n = t^{k_\circ}$, $y^n = t^{k_\bullet}$).

The category $\text{CM } C_k$ of f.g. $C_k$-modules free over $\mathbb{Z}$ is a full exact subcategory of $\text{MF}(Q; t)$ and $\nu : K(\text{CM } C_k) \xrightarrow{\cong} \mathbb{M}_k \subseteq \text{Wt } GL(n)$. $\text{CM } C_k$ contains $M(\mu)$ for all $\mu$ of chirality $k$; these are all the rk 1 modules (up to isom).

**Theorem** [JKS1] There is a cluster character $\Psi : \text{CM } C_k \to \mathbb{C}[\hat{\text{Gr}}_k^n]$ such that $\text{wt } \Psi_M = \nu_M$. In particular, $\Psi_M(\mu) = \Delta_J$.

Fact: $C_k$ is *thin*, i.e. each component $e_i C e_j$, for $i, j \in Q_0$, is a free $\mathbb{Z}$-module of rank 1. Hence, for all $i \in Q_0$, projectives $P_i = C e_i$ and (CM-)injectives $I_i = (e_i C)^\vee := \text{Hom}_\mathbb{Z}(e_i C, Z)$ are matching modules $M(J)$, in fact, for $J$ some cyclic interval.
Plabic graph $G$ and dual quiver with faces $Q$

Arrow directions follow strands. Boundary arrows and backwards paths in boundary faces are $x = x^\circ$ or $y = x^\bullet$ from orientation.

Dimer algebra $A$ has $\text{CM } A = \text{MF}(Q; t)$, with $K(\text{CM } A) = \mathbb{M}$ and all $\text{rk } 1$ modules are matching modules $M(\mu)$.

All $M \in \text{CM } A$ satisfy chirality relation $x^{k\bullet} = y^{k\circ}$ for some fixed $k$. 
Matchings on $G$ are cochains on $Q$ (Poincaré duality)

Since $|Q|$ is a disc, both horizontal sequences are exact and hence $\text{rank } M = \# Q_0$.

There is a **bdry value map** $\partial : M \rightarrow M_k$ (compatible with $\text{deg}$) that is dual to inclusion of chains: path $x^* \mapsto \sum \text{arrows in } x^*$.

Explicitly $\partial \mu = J = (J_\bullet, J_\circ)$, where $J_\bullet = \{ j \in C_1 : \mu(x^*_j) = 1 \}$.

The restriction $\rho_{AC} : CM A \rightarrow CM C$ categorifies $\partial$. 
Projectives and injectives

Consistency $\Rightarrow A$ is thin, so projectives $\mathcal{P}_i = Ae_i$ and injectives $\mathcal{I}_i = (e_iA)^\vee$ are matching modules $M(\mu)$ .. but which?

[Mu-Sp] define bases of matchings $m^{s/t}: \mathbb{Z}Q_0 \to \mathcal{M}$, whose bdry values $\partial m_j^{s/t}$ give source (s) and target (t) labellings for $G$.

**Prop [CKP]** For all $j \in Q_0$, we have $[\mathcal{P}_j] = m^s_j$ and $[\mathcal{I}_j] = m^t_j$. 

![Diagrams](image-url)
Boundary algebra, necklace and positroid

Bdry algebra $B = eAe$, where $e = \sum_{i \in C_0} e_i$ is bdry idempotent.

Restriction $\rho_{AC}$ factorises as $CM A \xrightarrow{\rho_{AB}} CM B \xrightarrow{\rho_{BC}} CM C$, where $\rho_{AB} : X \mapsto eX$ and $\rho_{BC}$ is a fully faithful embedding.

If $i \in C_0$, then $\rho_{AB} : Ae_i \mapsto Be_i$ and $(e_i A)^\vee \mapsto (e_i B)^\vee$, so these are the matching modules $M(N_i)$ and $M(N'_i)$ for necklace $N$ and reverse necklace $N'$.

In other words, the necklace is $B$.

Matching module $M(J)$ is in $CM B$ iff $J$ is in the positroid.
Projective resolution

Can view $m = m^s$ as the map $K(\mathcal{P}A) \xrightarrow{m} K(CM A)$ induced by inclusion of category $\mathcal{P}A$ of projective $A$-modules, thus $m^{-1}$ comes from projective resolution.

**Thm** [CKP] Each $M = M(\mu)$ in $CM A$ has a projective resolution

$$\bigoplus_{a \in \mu} A e_{ta} \rightarrow \bigoplus_{a \not\in \mu} A e_{ha} \rightarrow \bigoplus_{i \in Q_0} A e_i$$

[Ma-Sc] define weights for internal arrows $\text{wt}(a) \in \mathbb{Z}^{Q_0} = K(\mathcal{P}A)$.

**Cor** [CKP] For $\mu \in \mathbb{M}$,

$$m^{-1}(\mu) = \sum_{a \in Q_1}^{\text{ext}} \mu(a)[P_{ha}] + \deg(\mu) \sum_{i \in Q_0}^{\text{int}} [P_i] - \sum_{a \in Q_1}^{\text{int}} \mu(a)\text{wt}(a)$$

**Prop** [CKP] For all $j \in Q_0$, we have $m^{-1}(m^s_j) = [P_j]$. 
Newton-Okounkov cone

The restriction functor $\rho_{AB}: \text{CM } A \to \text{CM } B: X \mapsto eA \otimes_A X$ has a right adjoint $F: \text{CM } B \to \text{CM } A: M \mapsto \text{Hom}_B(eA, M)$.

Here the counit $\eta_X: X \to FeX$ is an embedding, i.e., if $eX = M$, then $X \subseteq FM$, so $FM$ is the maximal module which restricts to $M$.

For $M(J)$ in $\text{CM } B$, $FM(J)$ is a matching module $M(\mu)$ and $\mu$ is the minimal matching with $\delta \mu = J$ in the flip partial order.

Claim [JKS3] For $M \in \text{CM } C$, i.e. the $\hat{\text{Gr}}_k^n$ case, $z^{[FM]}$ is the leading monomial (a la [Ri-Wi]) in network coords of the clus. char. $\Psi_M$.

See [JKS2] for $\Psi_{M(J)} = \Delta_J$, which is given in network coords by the dimer partition function: $Z_J = \sum_{\mu: \delta \mu = J} z^\mu$

Expectation: (a) The set $\{[FM] \in \mathbb{M} : M \text{ in } \text{CM } C\}$ is precisely the integral points in the Ri-Wi Newton-Okounkov cone for $\hat{\text{Gr}}_k^n$.
(b) A basis of $\mathbb{C}[\hat{\text{Gr}}_k^n]$ is given by $\{\Psi_M : M \text{ general in } \text{CM } C\}$.
(c) Similar holds for positroid $\hat{\text{Gr}}_{\pi}$, by replacing $C$ by $B$. 
**Background: network torus and Muller-Speyer twist**

\[ \mathbb{M} \supseteq \text{deg}^{-1}(0) \cong \mathbb{Z}^Q_0 / c\mathbb{Z}, \] which is the character lattice of the usual network torus, in monodromy coordinates.

Thus \( \mathbb{M} \) is the character lattice of a torus \( \mathbb{M}^* \) that lifts the network torus to the positroid cone \( \hat{\text{Gr}}^o_\pi \), using the dimer part. fun.

\[
\mathbb{C} [\hat{\text{Gr}}^o_\pi] \to \mathbb{C} [\mathbb{M}^*]: \Delta J \mapsto \mathcal{Z}_J := \sum_{\mu: \delta \mu = J} z^\mu
\]

Note: for \( J \in \mathbb{N} \), i.e. \( \Delta J \) frozen, \( \mathcal{Z}_J \) is a monomial so invertible.

*Thm: [Mu-Sp]* There is an automorphism \( \tau: \hat{\text{Gr}}^o_\pi \to \hat{\text{Gr}}^o_\pi \) s.t.

\[
\mathbb{C} [(\mathbb{C}^*)^Q_0] \xrightarrow{(-m^{-1})^*} \mathbb{C} [\hat{\text{Gr}}^o_\pi] \xrightarrow{\tau^*} \mathbb{C} [\hat{\text{Gr}}^o_\pi]
\]

\[
\text{network} \quad \text{cluster}
\]
Application: Marsh-Scott twist

Rearrange the $m^{-1}$ formula, when $\mu$ is a (deg 1) matching to get

$$\text{wt}(\mu) - \text{wt}(G) = \sum_{a \in \partial \mu} [P_{ha}] - m^{-1}(\mu) \quad (*)$$

Recall: [Ma-Sc] define a twist $\sigma_\bullet: \hat{\text{Gr}}_k^n \rightarrow \hat{\text{Gr}}_k^n$ and prove that

$$\sigma_\bullet(\Delta_J) = Z_{J}^{MS} := z^{-\text{wt}(G)} \sum_{\mu: \partial \mu = J} z^{\text{wt}(\mu)} \quad \text{in cluster coords}$$

Define $p^\bullet: M_k \rightarrow \mathbb{Z}^{Q_0}: J \mapsto \sum_{a \in J} [P_{ha}]$.

Then $dmp^\bullet([M]) = [P^\bullet M] \in M_k$ for a projective cover $P^\bullet M \rightarrow M$.

$$(*) \Rightarrow Z_{J}^{MS} = z^{p^\bullet(J)} \sum_{\mu: \partial \mu = J} z^{-m^{-1}(\mu)}$$

Thm [CKP] For $M(J)$ in CM $C$, we have $\sigma_\bullet(\Delta_J) = \psi_{\Omega^\bullet M(J)}$, where $\Omega^\bullet M$ is the syzygy ker $P^\bullet M \rightarrow M$. 


