Dimeres and Webs.

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arXiv: 1404.3317

arXiv: 1506.00603 \{ Lecture 1 \}

arXiv: 1705.09424 (j. C. Fraser and I. Le) \ Lecture 2 \
Planar bipartite graph

Boundary vertices are black and degree one

\[ \Delta_I(N) = \sum_{\pi: \partial(\pi) = I} \text{wt}(\pi) \]

\[ |\pi| = k := \# \text{interior white} - \# \text{interior black} \]

\[ \partial(\pi) = \{1, 4\} \]

\[ \text{wt}(\pi) = a \]
Example of dimer generating function:

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\[ \Lambda_{13}(N) = ac + bd \]
Theorem 1

\[
(\Delta_i(N))_{I \in \mathcal{C}_k} \text{ defines a point in } Gr(k, n)
\]

ie. satisfies the Plücker relation

\[
\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1, i_2, \ldots, i_{k+1}} \Delta_{j_1, j_2, \ldots, j_{k+1}} = 0
\]

e.g. \( k = 2 \), \( i_1 = 1 \), \( j_1, j_2, j_3 = 2, 3, 4 \)

\[-\Delta_{12} \Delta_{34} + \Delta_{13} \Delta_{24} - \Delta_{14} \Delta_{23} = 0.\]

\[-a \cdot b + (ac+bd) \cdot 1 - a \cdot c = 0\]
Proof using “webs.”

**Definition:** A subgraph \( \Sigma \subset N \) is 2-weblike if it is a union of connected components:
1. paths between boundary vertices
2. interior cycles
3. single edges (called “doubled edge”)
such that every interior vertex is used.

\[
F_{c,T}(N) := \sum_{\Sigma} \text{wt}(\Sigma)
\]

- \( c \): partial non-crossing matching
- \( T \): boundary vertices used in doubled edges

\[
\text{wt}(\Sigma) = \prod_{ee \text{ path}} \text{wt}(e) \cdot \prod_{e \text{ edge}} \text{wt}(e)^2 \cdot 2^{\# \text{cycle}}
\]

Temperley–Lieb Immanant

\[
[ L, \text{ cf. Rhoades - Skandera} ]
\]

\[
[ \text{cf. Kenyon - Wilson} ]
\]
Double dimer

\[ I = 138 \]

\[ J = 379 \]
Definition $I, J \in \binom{n}{k}$ is compatible with $(\tau, T)$ if:

- $T = \text{i} \cap \text{j}$
- each chord of $\tau$ matches $i \in I - J$ with $j \in J - I$

Theorem 2 [cf. Kenyon-Wilson, Rhoades-Skandera, ...]

$$\Delta_I(N) \Delta_J(N) = \sum_{(\tau, T) \text{ comp with } (I, J)} F_{\tau, T}(N)$$

Example:

$\Delta_{124} \Delta_{356} = F_3^2 + F_5^2$

$F_{\tau, T}$ is a well-defined quadratic function on $\mathbb{C}[\tau(n)]$

- forms a cyclically invariant basis
- coincides with “dual canonical basis”, better than “standard monomial basis”

$$\Delta_{S_1} \Delta_{S_2}$$

$$\begin{bmatrix} S_1 & S_2 \end{bmatrix}$$

semi-standard
• **Theorem 2 ⇒ Theorem 1**

\[
\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1, \ldots, i_k, j_1, \ldots, j_k} \Delta_{j_1, \ldots, j_k} \quad \text{in terms of } \mathcal{F}_{\mathcal{I}}
\]

and do a sign-reversing involution.

• \( \Delta_I \Delta_J \leq \Delta_{\min}(I, J) \Delta_{\max}(I, J) \leq \Delta_{\text{sort}_1}(I, J) \Delta_{\text{sort}_2}(I, J) \)

\[ \begin{align*}
\Delta_{13567} \Delta_{23489} &\leq \Delta_{13467} \Delta_{23587} \leq \Delta_{13468} \Delta_{23579} \\
\text{sort} &= 1233456789
\end{align*} \]

[cf. Rhoades-Skandera, Faber-Postnikov, L.-Postnikov-Panyushev]
\( M(N) := \left\{ I \in \binom{[n]}{\leq k} \mid \Delta_I(N) \neq 0 \right\} \) matroid of \( N \).

\( I, J \in M \Rightarrow \text{sort}_1, \text{sort}_2 \in M \) [L. Postnikov]

\( M^{(2)}(N) := \left\{ \tau, \tau \right\} \mid F_{\tau, \tau}(N) \neq 0 \) "quadratic matroid"

• any positive quadratic function, homogeneous with respect to torus action, is a positive linear combination of \( F_{\tau, \tau} \)

• applications to dimers?
Based on joint work with C. Fraser and I. Le

Webs

\[ \{ [v_1, v_2, \ldots, v_n] \} \]

\( \text{Gr}(k, n) = R(k, n) = \text{SL}_k \text{ invariant functions on } \mathbb{C}^k \times \cdots \times \mathbb{C}^k \) / relations

\[ \det(v_1 v_2) \]

\[ \Lambda^2 \mathbb{C}^3 \to \mathbb{C}^3 \otimes \mathbb{C}^3 \]

\[ \text{Reshetikhin-Turaev} \]

\[ \text{Kuperberg} \]

\[ e \in R(3, q)_3 \]

\[ v_i \in \mathbb{C}^3 \]
Clusters and webs

Cluster structure of $\mathcal{O}[Gr(3,n)]$ (+ more) can be described in terms of web invariants:

- initial cluster, quivers, coefficient
- cluster variables are indecomp. web invariants
- monomial, tensor invariant
- compatibility

[Fraser, Gr(3,9)]

Conjecture. [Fomin-Polyansky]

$\{\text{more}\}$
SL(r)-webs

untagged $SL(5)$-web

tagged $SL(5)$-web

$\mathbf{c}$
\[ \Lambda^a(\mathbb{C}^r) \otimes \Lambda^b(\mathbb{C}^r) \otimes \Lambda^c(\mathbb{C}^r) \rightarrow \Lambda^{a+b+c}(\mathbb{C}^r) \]

Wedge
\[ x_1 \otimes x_2 \otimes x_3 \rightarrow x_1 \wedge x_2 \wedge x_3 \]

Shuffle
\[ x_1 \wedge \ldots \wedge x_{a+b+c} \rightarrow \sum \pm (x_{i_1} \wedge \ldots \wedge x_{i_a}) \otimes (\ ) \otimes (\ ) \]

\[ \Lambda^a(\mathbb{C}^r) \otimes \Lambda^{r-a}(\mathbb{C}^r) \rightarrow \Lambda^r(\mathbb{C}^r) \cong \mathbb{C} \]

\[ \Lambda^r(\mathbb{C}^r) \rightarrow \Lambda^a(\mathbb{C}^r) \otimes \Lambda^{r-a}(\mathbb{C}^r) \]
Lemma. Any tagging $\hat{W}$ of a web $W$ produces the same function up to sign.

$$W_\lambda(r) = \text{Hom}_{\text{SL}(r)}(\otimes^\lambda \mathbb{C}^r, \mathbb{C}) \quad \lambda = (\lambda_1, \ldots, \lambda_n)$$

$$W(r, n) = \text{Hom}_{\text{SL}(r)}(\otimes \mathbb{C}^r, \mathbb{C})$$

are spanned by appropriate $\hat{W}$.

Ex. $r = 1$

$$\alpha_i \in \mathbb{C}$$

$$\hat{W} = \pm \alpha_1 \alpha_2 \alpha_3 \alpha_4$$
\( r = 2 \)

\[
v_i \in \mathbb{C}^2
\]

\[
= \pm \det(v_1 | v_4) \det(v_2 | v_3)
\]

basis of \( W(2,2n) \) = noncrossing matchings on \([2n]\)

\[\text{[Rumer-Teller-}\] Weyl\]

\[
= \text{trace}(\text{Id}: \mathbb{C}^2 \rightarrow \mathbb{C}^2)
\]

\[
= \pm 2
\]
\[ r = 3 \]

\[ v^6 = 3 \]

\[ v^2 = 2 \]

\[ v^1 = 1 + 3 \]

\[ \text{[Kuperberg] basis = irreducible/elliptic webs} \]

ie. no contractible loops, no square face
Definition An $r$-weblike subgraph $W \subset N$ is a subgraph using all vertices of $N$, with each edge labeled by a multiplicity $m(e)$, such that

$$\sum_{e \text{ incident to } v} m(e) = r \quad \text{for interior } v$$

$$\text{wt}(W) = \prod_{e} \text{wt}(e)^{m(e)}$$

$$\lambda(W) = \text{boundary multiplicities}$$

Each such $W$ gives an untagged $SL(r)$-web and a tensor invariant $\hat{W} (= \pm \hat{W} \text{ for some tagging})$
Theorem 1 If $\Delta_I(N) = \Delta_I(N')$ for all $I$, then

$$\text{Web}_r(N; \lambda) = \text{Web}_r(N'; \lambda) \in \mathcal{W}_\lambda(r)$$

for all $r, \lambda$. 

[Fraser, L., Le]
\[ C[\text{Gr}(k,n)]_{124} \] be the \( \lambda \)-weight component.

\[ \Delta_{124}A_{246} \] has weight \((1,2,0,2,0,1)\)

**Theorem 2** \[ \text{Imm} : \omega_1(U) \to C[\text{Gr}(k,n)]_{124} \]

\[ \varphi \to (N \to \varphi(\text{Web}_r(N;\lambda))) \]

is an isomorphism.

**Theorem 3** \( \Delta_{I_1}(N) \Delta_{I_2}(N) \ldots \Delta_{I_r}(N) = \text{sign}(S)\text{ Web}_r(N;\lambda) \bigg|_{c^S} \)

where

\[ S = (S(1), \ldots, S(n)) \quad S(j) \subseteq \{i \in [r] \} \]

\[ c^S : e_5 := e_{S(1)} \otimes \ldots \otimes e_{S(n)} \in \Lambda^I(C^r) \otimes \ldots \otimes \Lambda^I(C^r) \]

I, S related by

\[ (e_{124} = e_1 \lambda e_2 \lambda e_4) \]

\[ I_i = \{ j \in [n] : i \in S(j) \} \]

ie. transpose as 01-matrices.
Remarks

- $r=2$

$\mathrm{W}_1(2)$ has noncrossing matching basis:

\begin{align*}
e \in \mathrm{W}_{(1,1,1,1)}(2) = \mathrm{W}(2,6) & \leftrightarrow \quad \mathbb{C}[G(3,6)] \\
\mathrm{Imm}(\text{dual basis}) & = F_{2,1}.
\end{align*}
• $r=3$

[Kuperberg] $W_3(3)$ has basis of irreducible/elliptic webs.

• For $r>3$, no generally agreed upon web basis is known. There is a dual canonical basis.
Generators for relations b/w webs are known

\[ \text{[Kuperberg, ... , Cautis - Kamnitzer - Morrison]} \]

Theorem: Under \( \text{Web}_N(N; t) = \text{Web}_N(N'; t) \)

Local moves for planar bipartite graphs \( \Rightarrow \) web relations

Square move for tagged webs
Further directions

- Quantization?

- Positivity of $W$ as function on $Gr(r,n)$?

- Applications to dimer?

- Relation Imm to cluster structure?