The aim of this note is to prove the following result:

Theorem 1. *Let $x_1(t)$, $x_2(t)$, …, $x_n(t)$ be smooth positive real valued functions. Let e_k be the k-th elementary symmetric function in the x_i's. Suppose that $\frac{d}{dt} e_k(x_1(t), \dots, x_n(t)) > 0$ for $1 \leq k \leq n-1$ and $e_n(x_1(t), \dots, x_n(t))$ is constant. Then $\frac{d}{dt} \sum (\log x_i(t))^2$ is positive.*

We begin with some computations whose connection to the previous problem will not be immediately clear.

Lemma 2. *Let x_1, x_2, …, x_n be distinct positive real numbers. Define*

$$h(r) = \sum_i \frac{x_i^r}{\prod_{a \neq i}(x_i - x_a)}.$$

Then $h(r)$ has simple zeroes at 0, 1, …, $n-2$ and no other zeroes. We have $h(r) > 0$ for sufficiently positive r.

Proof. Since $h(r)$ is a linear combination of n functions of the form $c \cdot x^r$, by Descartes' rule of signs for real exponents, h has at most $n-1$ roots counted with multiplicity. So it is enough to show that $h(k) = 0$ for $k = 0$, 1, …, $n-2$. For any integer j, we note that $h(k)$ is a rational function in x_1, x_2, …, x_n. This rational function is homogenous of degree $k - n + 1$. But we also note that, for $k \geq 0$, this rational function is a polynomial. The only possible poles of h are along $x_i - x_j$, but h does not blow up along these hyperplanes. So, for k an integer between 0 and $n-2$, the function $h(x_1, \dots, x_n)$ is a rational function of negative degree, and hence zero.

Finally, we must confirm the sign of h for large r. Without loss of generality, let $x_1 > x_2$, x_3, …, x_n. For r large, the x_1^r term dominates all others, and it has positive sign. ☐

Remark 3. *For any integer k, the function $h(k)$ is the complete homogenous symmetric polynomial of degree $k - n + 1$.*

Corollary 4. *For distinct positive real numbers x_1, x_2, …, x_n and $k = 0$, 1, …, $n-2$, we have*

$$(-1)^{n-2-k} \sum_i \frac{x_i^k \log x_i}{\prod_{a \neq i}(x_i - x_a)} > 0.$$

Proof. The sum is $h'(k)$. Since we know the sign of h on each of $(-\infty, 0)$, $(0,1)$, …, $(n-3, n-2)$, $(n-2, \infty)$, we know the sign of h' at the boundary points. ☐

Next, we perform some computations with matrices of symmetric functions:

Lemma 5. *Let x_1, …, x_n be distinct positive reals. Let J be the Jacobian matrix $J_{ij} = \partial e_j / \partial x_i$. Set $u_i = \prod_{a \neq i}(x_i - x_a)$. Then*

$$J^{-1} = \begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} \\ -x_1^{n-2} & -x_2^{n-2} & \cdots & -x_{n-1}^{n-2} & -x_n^{n-2} \\ & & \ddots & & \\ (-1)^{n-2} x_1 & (-1)^{n-2} x_2 & \cdots & (-1)^{n-2} x_{n-1} & (-1)^{n-2} x_n \\ (-1)^{n-1} & (-1)^{n-1} & \cdots & (-1)^{n-1} & (-1)^{n-1} \end{pmatrix} \begin{pmatrix} u_1^{-1} & & & & \\ & u_2^{-1} & & & \\ & & \ddots & & \\ & & & u_{n-1}^{-1} & \\ & & & & u_n^{-1} \end{pmatrix}$$

Proof. Note that $\partial e_i / \partial x_j$ is $e_{i-1}(x_1, x_2, \dots, \widehat{x_j}, \dots, x_n)$, so these are the entries of J.

Let V be the first matrix on the right hand side and let U be the diagonal matrix. We want to prove $J^{-1} = VU$ or, equivalently, $U^{-1} V^{-1} = J$. The matrix V is known as the

Vandermonde matrix, and the formula for the inverse of the Vandermonde matrix is well known. Manipulating that formula, and the description of the entries of J above, the result follows. $\square$

Define $f(x_1, \ldots, x_n) = \sum (\log x_i)^2$.

Lemma 6. *For any distinct positive reals $x_1, \ldots, x_n$, we can write the vector ∇f as a linear combination $\sum c_j \nabla e_j$. Moreover, the coefficients $c_1, c_2, \ldots, c_{n-1}$ are positive.*

Proof. The vectors ∇e_j are the columns of J. Lemma 5 gives an inverse to J, so some such scalars c_j exist, and they are the entries of the vector $-J^{-1}\nabla f$. Plugging in the formula from the lemma, we obtain

$$c_j = (-1)^{j-1} \sum_i \frac{x_i^{n-j}}{u_i} \frac{\partial f}{\partial x_i} = (-1)^{j-1} \sum_i \frac{2x_i^{n-j-1} \log x_i}{\prod_{a \neq i}(x_a - x_i)}.$$

When $j = 1, 2, \ldots, n-1$, we have $n - j - 1 = 0, 1, \ldots, n-2$. By Corollary 4, the sum on the right has sign $(-1)^{n-2-(n-j-1)} = (-1)^{j-1}$, so $c_j > 0$. $\square$

Now, suppose we have real functions $x_1(t), x_2(t), \ldots, x_n(t)$ as in Theorem 1. Set $\gamma = (\partial x_1/\partial t, \partial x_2/\partial t, \ldots, \partial x_n/\partial t)$. Then the dot products $(\nabla e_1) \cdot \gamma$, $(\nabla e_2) \cdot \gamma$, $\ldots$, $(\nabla e_{n-1}) \cdot \gamma$ are all positive and $(\nabla e_n) \cdot \gamma = 0$. It follows from Lemma 6 that $(\nabla f) \cdot \gamma > 0$, as desired.