Lecture I: Introduction to Tropical Geometry
David Speyer
The field of Puiseux series

\( \mathbb{C}[[t]] \) is the ring of formal power series \( a_0 + a_1 t + \cdots \) and \( \mathbb{C}((t)) \) is the field of formal Laurent series: \( a_{-N} t^{-N} + a_{-N+1} t^{-N+1} + \cdots \).

\[
\mathcal{K} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n})), \quad \mathcal{R} = \bigcup_{n \geq 1} \mathbb{C}[[t]].
\]

\( v : \mathcal{K}^\times \to \mathbb{Q}, \quad v \left( \sum a_i t^{i/N} \right) = \min \left( \frac{i}{n} : a_i \neq 0 \right). \)

We should think of \( \mathcal{R} \) as functions of \( t \) in some small neighborhood \( [0, \epsilon) \) and \( \mathcal{K} \) as functions on \( [0, \epsilon) \) with some pole.

If the relevant sums converge, then

\[
v(f) = \lim_{t \to 0^+} \frac{\log f(t)}{\log t}.
\]

\( \mathcal{K} \) is conveniently algebraically closed.
For $X$ in $(\mathcal{K}^\times)^n$, set $\text{Trop } X$ to be $v(X) \subseteq \mathbb{Q}^n$. You’ll also see $\overline{v(X)} \subseteq \mathbb{R}^n$. Each of these creates some notational awkwardness at the beginning, but we will soon have theorems telling us not to worry about it.

Everyone’s first example: $x + y + 1 = 0$

If $x(t) + y(t) + 1 = 0$, then either

• $v(x) \geq 0$ and $v(y) = 0$.
• $v(y) \geq 0$ and $v(x) = 0$.
• $v(x) = v(y) \leq 0$.

This example has “constant coefficients”, meaning that there are no $t$’s in the equation $x + y + 1 = 0$. We’ll stick to constant coefficient examples for a while.
We’ll talk about

• Hypersurfaces with constant coefficients
• The “initial variety” construction
• General varieties with constant coefficients
• Nonconstant coefficients
\[ \text{Trop } X = \lim_{t \to 0^+} \frac{\log |X|}{\log t}. \]

\[ \log t = -2 \quad \log t = -5 \quad \log t = -10 \quad t = 0 \]
Let \( F \in \mathbb{C}[x_1^\pm, x_2^\pm, \ldots, x_n^\pm] \) and let \( X \) be the hypersurface \( F = 0 \). Let \( \Delta(F) \) be the Newton polytope

\[
\Delta(F) = \text{Hull}(a \in \mathbb{Z}^n : x^a \text{ has nonzero coefficient in } F) \subset \mathbb{R}^n
\]

For any \( w \in \mathbb{R}^n \), the function \( \langle w, \rangle \) is minimized on a face of \( \Delta(F) \). Divide \( \mathbb{R}^n \) up into cones according to which face of \( \Delta(F) \) this minimum occurs on. This is the normal fan to \( \Delta(F) \).

**Theorem:** Trop \( X \) is the union of the codimension one faces of the normal fan.

Trop \( X \) is a bunch of rational polyhedral cones. This is why we don’t care much about the difference between \( \mathbb{Q} \) and \( \mathbb{R} \).
Is there a geometric meaning to this normal fan construction? Yes!

$F = \sum F_a x^a$ continues to have constant coefficients. Consider $w \in \mathbb{Q}^n$ and let $\Gamma$ be the face of $\Delta(F)$ where $\langle w, \rangle$ is minimized. Let $\text{in}_w F$ be $\sum_{a \in \Gamma} F_a x^a$. 

![Diagram](image)
Let \( w = (w_1, \ldots, w_n) \in \mathbb{Q}^n \). Let \( \text{in}_w X \) be the hypersurface cut out by \( \text{in}_w X \). There is a point of \( X \) of the form 
\[
(a_1 t^{w_1} + \cdots, a_2 t^{w_2} + \cdots, \ldots, a_n t^{w_n} + \cdots)
\]
if and only if 
\[
(a_1, \ldots, a_n) \in \text{in}_w(X).
\]

In particular, this explains why \( \text{Trop} X \) is the \( w \) for which \( \text{in}_w F \) is not a monomial.
What happens when $X$ is not a hypersurface?

$$\text{Trop } X = \bigcap_{F|_X = 0} \text{Trop}\{F = 0\}$$

Moreover, there is a finite set of polynomial $F_1, F_2, \ldots, F_r$ such that

$$\text{Trop } X = \text{Trop}\{F_1 = 0\} \cap \text{Trop}\{F_2 = 0\} \cap \cdots \cap \text{Trop}\{F_r = 0\}$$

Therefore, $\text{Trop } X$ is a union of finitely many rational cones. (Rational means of the form $\{w : \langle w, a_1 \rangle, \langle w, a_2 \rangle, \ldots, \langle w, a_r \rangle \geq 0\}$ for $a_i$ vectors in $\mathbb{Z}^n$.) This is why we don’t have to worry much about the difference between $\mathbb{Q}$ and $\mathbb{R}$. 
The variety $in_w X$ is cut out by $in_w F$, for $F$ in $I$. Again, there is a point of $X$ of the form $(a_1 t^{w_1} + \cdots, a_2 t^{w_2} + \cdots, \ldots, a_n t^{w_n} + \cdots)$ if and only if $(a_1, \ldots, a_n) \in in_w (X)$. And we can choose the finite generating set $F_1, F_2, \ldots, F_r$ on the previous slide so that $in_w F_1$, $in_w F_2$, $\ldots$, $in_w F_r$ cut out $in_w (X)$ for all $w$.

We have $\dim X = \dim \text{Trop } X$.

If $X$ is connected in codimension 1, so is $\text{Trop } X$.

Near a point $w \in \text{Trop } X$, $\text{Trop } X$ looks like a translate of $\text{Trop in}_w X$. 
What about nonconstant coefficients? I don’t expect to use these in my lectures, but I certainly expect professors Mikhalkin and Gross will.

So, what does $\text{Trop}(xy + x + y + t)$ look like? It is the union of the following possibilities:

- $v(y) = 0$, $v(x) \leq 0$.
- $v(x) = 0$, $v(y) \leq 0$.
- $0 \leq v(x) = v(y) \leq 1$.
- $v(y) = 1$, $v(x) \geq 1$.
- $v(x) = 1$, $v(y) \geq 1$. 
Once again, we can see this as \( \lim_{t \to 0^+} \frac{\log |X(t)|}{\log t} \).

\[
\begin{align*}
  t &= 0.5 & t &= 0.1 & t &= 0.01 & t &= 0
\end{align*}
\]
Let $F = \sum F_a x^a$ be a polynomial in $K[x_1^\pm, \ldots, x_n^\pm]$, where $a$ ranges over some finite subset $A$ of $\mathbb{Z}^n$. Let $w \in \mathbb{Q}^n$.

Let

$$u = \min_{a \in A} v(F_a) + \langle a, w \rangle.$$

Let $F_a = g_a t^{u - \langle a, w \rangle} + \text{higher order terms}$, so $g_a \neq 0$ if and only if the minimum defining $u$ is achieved at $a$. Set

$$\text{in}_w F = \sum g_a x^a$$

and

$$\text{in}_w X = \{ \text{in}_w F = 0 \}.$$

There is a point of $X$ of the form

$$(a_1 t^{w_1} + \cdots, a_2 t^{w_2} + \cdots, \ldots, a_n t^{w_n} + \cdots)$$

if and only if

$$(a_1, \ldots, a_n) \in \text{in}_w (X).$$

So $w \in \text{Trop } X$ if and only if $\text{in}_w F$ is not a monomial.
Combinatorially, this involves working with a convex subdivision of the Newton polytope of $F$. 
For nonhypersurfaces, let $I \subset \mathcal{K}[x_1^\pm, \ldots, x_n^\pm]$ be the ideal of $X$. Define $i_w X$ to be \{x : (i_w F)(x)\} = 0, for $F \in I$. Again, we can find a finite subset $G$ of $I$ such that $i_w X$ is generated by $(i_w F)_{F \in G}$ for all $w$.

There is a point of $X$ of the form $(a_1 t^{w_1} + \cdots, a_2 t^{w_2} + \cdots, \ldots, a_n t^{w_n} + \cdots)$ if and only if $(a_1, \ldots, a_n) \in i_w (X)$. So $w \in \text{Trop} X$ if and only if $i_w F$ is not a monomial.

Again, $\text{Trop} X$ is a finite rational polyhedral complex, of dimension $\dim X$. One can recover the degree of $X$, and limited information about the cohomology of $X$, from $\text{Trop} X$. 