Linear Recurrences and Rational Generating Functions

Today we talk about an class of generating functions which are basic, but very important.

Let me start with a key identity, which we will use over and over again: For any complex number r, we have

$$\frac{1}{1-rx} = 1 + rx + r^2x^2 + r^3x^3 + \cdots = \sum_{n=0}^{\infty} r^n x^n.$$

This is, of course, just the geometric series. You'll be seeing a lot of it.

Warm up problem: Let a_i be given by the recursion

$$a_n = a_{n-1} + 6a_{n-2} \quad a_1 = 1 \quad a_0 = 0.$$

So the first few a's are 0, 1, 1, 7, 13, 55, 133, ...

What is a closed formula for $A(x) := \sum a_n x^n$? What about for a_n?

There is a general recipe to follow here.

Step One: Turn the recursion into an equation obeyed by $A(x)$.

$$\begin{array}{rcllllllllll} A(x) & = & x & + & x^2 & + & 7x^3 & + & 13x^4 & + & 55x^5 & + \cdots \\ & = & x & & & & & & & & & \\ & & & + & (\, x^2 & + & x^3 & + & 7x^4 & + & 13x^5 & + \cdots) \\ & & & & & + & 6\,(x^3 & + & x^4 & + & 7x^5 & + \cdots) \end{array}$$

so

$$A(x) = x + xA(x) + 6x^2 A(x).$$

Be careful with those low degree terms!

Step Two: Find $A(x)$. Trivial in this case:

$$A(x) = \frac{x}{1 - x - 6x^2}.$$

Step Three: If you want a formula for a_n, expand $A(x)$ in partial fractions. I say "if" because, often, having a formula for $A(x)$ is more useful than a formula for a_n. That said, if we want a formula for a_n, we can get one:

$$A(x) = \frac{x}{1 - x - 6x^2} = \frac{1/5}{1 - 3x} - \frac{1/5}{1 + 2x}.$$

(Remember partial fractions?)

So

$$A(x) = \frac{1}{5}\left(1 + 3x + 6x^2 + 27x^3 + \cdots\right) - \frac{1}{5}\left(1 - 2x + 4x^2 - 8x^3 + \cdots\right)$$

and

$$a_n = \frac{1}{5}\left(3^n - (-2)^n\right).$$

Let's check a few values. $7 = (27+8)/5$, $13 = (81-16)/5$, $55 = (243+32)/5$. Looks good!

A few notes: If you have multiple roots, you'll get terms like $1/(1 - rx)^k$. Those contribute

$$\frac{1}{(1 - rx)^k} = 1 + krx + \binom{k+1}{2} r^2x^2 + \binom{k+2}{3} r^3x^3 + \cdots.$$

So these sorts of terms will get you polynomials times exponentials, as opposed to pure exponentials.

If you learned partial fractions in a typical calculus course, you probably learned that you get some constant/linear terms and some linear/quadratic terms. This is becuase the American calculus curriculum, for some reason, is terrified of complex numbers. If you have $1/(1 + 4x + 13x^2)$, just factor $1 + 4x + 13x^2 = (1 + (2 + 3i)x)(1 + (2 - 3i)x)$ and go from there!

One more example of this form: The fibonacci numbers are given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. You've probably seen these: 1, 1, 2, 3, 5, 8, 13, 21, ... Leaving the details to you, we have

$$F(x) = x + xF(x) + x^2F(x)$$

so

$$F(x) = \frac{x}{1 - x - x^2}.$$

Set τ_1 and τ_2 equal to $(1 \pm \sqrt{5})/2$, so $1 - x - x^2 = (1 - \tau_1 x)(1 - \tau_2 x)$. Then

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \tau_1 x} - \frac{1}{1 - \tau_2 x} \right) = \frac{1}{\sqrt{5}} \sum (\tau_1^n - \tau_2^n) x^n$$

and

$$F_n = \frac{\tau_1^n - \tau_2^n}{\sqrt{5}}.$$

This example points out that irrational numbers are likely to turn up in these computations, and that they cause no special problems.

Let's summarize the result of these computations:

Theorem: Let a_n be a sequence of complex numbers, and set $A(x) = \sum a_n x^n$. The following are equivalent:

1. There is a recursion

   $$a_n + q_1 a_{n-1} + q_2 a_{n-1} + \cdots + q_k a_{n-k} = 0$$

   which holds for sufficiently large n.

2. The generating function $A(x)$ is a rational function $P(x)/Q(x)$ for some polynomials $P(x)$ and $Q(x)$.

3. There are rational numbers α_1, α_2, ..., α_r and polynomials $f_1(n)$, $f_2(n)$, ..., $f_r(n)$ such that, for n sufficiently large,

   $$a_n = \sum f_i(n) \alpha_i^n.$$

It is worth knowing how the above concepts connect. The roots of $Q(x)$ are α_i^{-1}, and $\deg f_i$ is the multiplicity of the root α_i^{-1}. The polynomial $Q(x)$ is $1 + q_1 x + q_2 x^2 + \cdots + q_n x^n$. Also, (1) holds for $n \geq k$ if and only if (3) holds for all n if and only if $\deg P < \deg Q$.

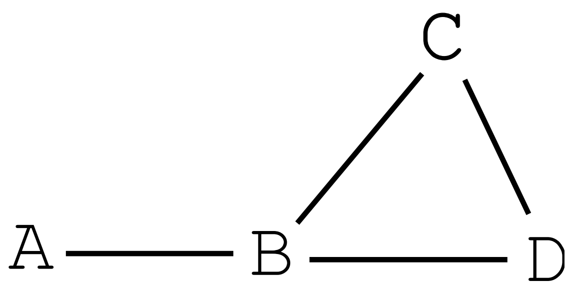
Asymptopics: The rate of growth of a_n is like α^n where α^{-1} is the smallest root of $Q(x)$. If α has multiplicity k, then the rate of growth is $\approx n^k \alpha^n$.

Some easy corollaries: Suppose that a_n and b_n obey linear recurrences. Then so do the sequences $a_n + b_n$, a_{kn}, $(a_0 + a_1 + \cdots + a_n)$ and $a_n b_n$. In the first three cases, it is easy to describe the corresponding operation on rational generating functions; in the last case, there is no simple description.

Let's see some places where rational generating functions show up:

Walks in graphs: Let G be any graph with finitely many vertices. Consider the number of length n paths from vertex i to vertex j. These will always obey linear recurrences. Let's see an example:

The following problem is taken from a blackboard in *Good Will Hunting.*



How many paths are there, are length n, from vertex A to itself?

It's best to approach this more generally. Let a_n, b_n, c_n and d_n be the number of paths from A to A, B, C and D respectively, with the generating functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$. Then

$$\begin{aligned}
A(x) &= 1 + xB(x) \\
B(x) &= xA(x) + xC(x) + xD(x) \\
C(x) &= xB(x) + xD(x) \\
D(x) &= xB(x) + xC(x)
\end{aligned}$$

We could solve these linear equations as they are, but it will be prettier to write things in terms of matrices.

$$\begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}.$$

So

$$\begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix} = \left(\mathrm{Id} - x \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Without inverting the whole matrix, let's see what the denominator will be. Clearly, it will be the determinant

$$\det \left(\mathrm{Id} - x \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \right) = 1 - 4x^2 - 2x^3 + x^4.$$

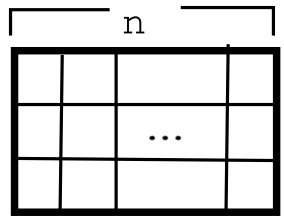
So the rate of growth of a_n will be roughly α^n where $\alpha \approx 2.17$ is the largest root of $y^4 - 4y^2 - 2y + 1$. Note that the α_i's are also the eigenvalues of the adjacency matrix.

Words formed by concatenating strings from a finite list: How many words are there of length n of the form $LALALAALAALALALAA$. Well, the generating function is $1/(1 - x^2 - x^3)$ – do you see why? If we keep track of the number of L's and A's separately, then we get $1/(1 - yz - yz^2)$. So the number of such words is roughly β^n, where $\beta \approx 1.32$ is the largest root of $y^3 - y - 1 = 0$.

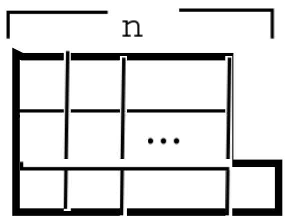
For the computer scientists in the audience, the more general statement is that we can count words corresponding to any rational expression.

The transition matrix method: How many ways can we tile a $3 \times n$ rectangle with dominoes?

Call this number a_n. The best way to answer this question is to simultaneously anwser all the similar questions. Let b_n, c_n, d_n and e_n be the number of ways to tile the similar shapes:



A



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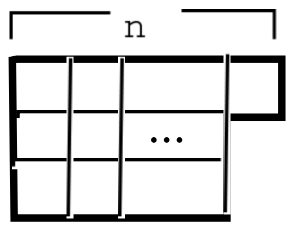

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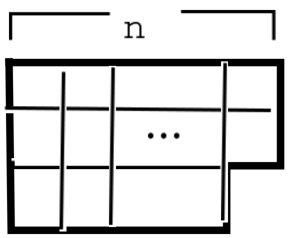

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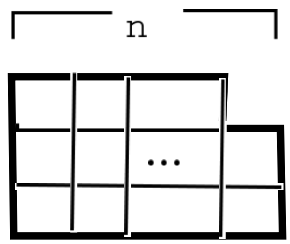

E

Then

$$\begin{aligned} a_n &= a_{n-2} + d_{n-1} + e_{n-1} \\ b_n &= e_{n-1} \\ c_n &= d_{n-1} \\ d_n &= a_{n-1} + c_{n-1} \\ e_n &= a_{n-1} + b_{n-1} \end{aligned}$$

We can clean up the computations by using a little thought: we have $b_n = c_n$ and $d_n = e_n$ by symmetry. And, using the second and third equation, we can eliminate b_n and c_n to give

$$\begin{aligned} a_n &= a_{n-2} + 2d_{n-1} \\ d_n &= a_{n-1} + d_{n-2} \end{aligned} .$$

We get

$$\begin{pmatrix} A(x) \\ D(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} x^2 & 2x \\ x & x^2 \end{pmatrix} \begin{pmatrix} A(x) \\ D(x) \end{pmatrix}.$$

So

$$\begin{pmatrix} A(x) \\ D(x) \end{pmatrix} = \begin{pmatrix} 1 - x^2 & -2x \\ -x & 1 - x^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Doing the computation, it turns out

$$A(x) = \frac{1 - x^2}{1 - 4x^2 + x^4}.$$

Quick quiz: why do only even powers of x occur?