We are finally ready to return to our goal of studying solvability of equations. Suppose that we start with a polynomial \( f(x) \) whose coefficients are in a field \( F_0 \). We compute a sequence of numbers \( \theta_1, \theta_2, \theta_3, \ldots \) where each number is either in our starting field \( F_0 \), or is computed from earlier numbers using the functions +, −, ×, ÷, \( \sqrt[n]{\cdot} \). Let \( F_i = F_0[\theta_1, \theta_2, \ldots, \theta_i] \). So, if \( \theta_i \) is computed using +, −, ×, ÷ then \( F_i = F_{i-1} \) and, if \( \theta_i \) is computed using \( \sqrt[n]{\cdot} \), then \( F_i = F_{i-1}[\sqrt[n]{a}] \) for some \( a \) in \( F_{i-1} \) and some positive integer \( m \). We want to study whether we can ever get to a root of \( f(x) \) by this process.

We now state, precisely, what we will prove:

**Theorem:** Let \( F_0 \) be a field, let \( f(x) \) be a polynomial with coefficients in \( F_0 \) and let \( K \) be a splitting field for \( f(x) \). Suppose that \( \text{Aut}(K/F_0) \) is \( S_n \), for \( n \geq 5 \). Let \( F_0, F_1, F_2, \ldots \), be a sequence of fields such that each \( F_{i+1} \) is of the form \( F_i[\sqrt[m_i]{a_i}] \) for some positive integer \( m_i \) and some \( a_i \) in \( F_i \). Then \( f(x) \) does not have roots in any of the \( F_i \).

Suppose that we did have fields as above and \( f(x) \) has a root in \( F_r \). In this worksheet, we will introduce a lot of fields. The diagram above shows the relation between them. The solid lines are the hypotheses of the theorem, and the dashed lines are the containments you will be proving:

Let \( M \) be the LCM of all the \( m_i \). Let \( g_i(x) \) be the minimal polynomial of \( a_i \) over \( F_0 \). Let \( L_i \) be the splitting field of \((x^M - 1) \prod_{i=0}^{j-1} g_i(x^{m_i}) \) over \( F_0 \). Notice that \( L_0 \) is the splitting field of \( x^M - 1 \) over \( F_0 \).

**Problem 17.1.**
1. Show that we can embed \( F_i \) into \( L_i \), as in the diagram.
2. Show that we can embed \( K \) into \( L_r \), as in the diagram.

**Problem 17.2.**
1. Show that \( g_i(x) \) splits in \( L_i \). Let its roots be \( \alpha_1, \alpha_2, \ldots, \alpha_s \).
2. Show that \( L_{i+1} = L_i[\sqrt[m_i]{\alpha_1}, \sqrt[m_i]{\alpha_2}, \ldots, \sqrt[m_i]{\alpha_s}] \).

Inside \( \text{Aut}(L_r/F_0) \), we have the subgroups \( G_i = \text{Aut}(L_r/L_i) \). Put \( \text{Aut}(L_r/F_0) = G \).

**Problem 17.3.**
1. Show that we have a surjection \( G \to \text{Aut}(K/F_0) \).
2. Show that we have a short exact sequence \( 1 \to G_i \to G_{i-1} \to A_i \to 1 \) where \( A_i \) is abelian.
3. Show that we have a short exact sequence \( 1 \to G_0 \to G \to B \to 1 \) where \( B \) is abelian.