The aim of this worksheet is to prove, in a certain sense, that there is no universal formula for the roots of a quintic equation. We first lay out exactly what we will, and will not, prove.

Let $L$ be the field of formal rational functions in the variables $r_1, r_2, r_3, r_4$ and $r_5$. Let $K \subset L$ be the field of symmetric rational functions in these variables. Thus, if we start with a general quintic $x^5 - e_1 x^4 + e_2 x^3 - e_3 x^2 + e_4 x + e_5 = (x - r_1)(x - r_2) \cdots (x - r_5)$, then the coefficients $e_1, \ldots, e_5$ are in $K$.

**Theorem:** Suppose we compute a sequence of elements of $L$, starting with $e_1, e_2, \ldots, e_5$, and applying the operators $+,-,\times,\div$ and $\sqrt[n]{}$. Every time that we apply the operator $\sqrt[n]{}$, we insist that the output of the $n$-th root still lies in $L$. Then we can never obtain the elements $r_1, r_2, \ldots, r_5$ of $L$.

We note what we are not proving. We are not considering the possibility of formulas where the $n$-th root leaves the realm of rational functions. Even more so, we cannot take any particular quintic with rational coefficients and conclude that its roots cannot be expressed in terms of $+, -, \times, \div$ and $\sqrt[n]{}$. In the second half of the course, we will address these issues.

But this Theorem is already strong enough to rule out a formula that looks anything like the quadratic, cubic or quartic formulas. We now turn to the proof.

Define $F \subset L$ to be the field of functions which are fixed by the alternating group $A_5 \subset S_5$. We will show that the operations $+,-,\times,\div$ and $\sqrt[n]{}$ cannot get us out of $F$.

**Problem 9.1.** Suppose that $f(r_1, r_2, \ldots, r_5)$ is in $F$ and $\sqrt[n]{f}$ is in $L$. Show that $\sqrt[n]{f}$ is in $F$.

**Problem 9.2.** Suppose that $f(r_1, r_2, \ldots, r_5)$ is in $F$, that $\sqrt[n]{f}$ is in $L$ and that $f \neq 0$. Show that $\chi(w) := w(\sqrt[n]{f})$ is a character of $A_5$. (Some people found this phrasing confusing. Here is an alternate phrasing which I would find more confusing, but maybe you won’t: Let $f \in L$ be nonzero and suppose that there exists a function $h \in F$ with $h^n = f$. Define $\chi(w) = \frac{w(h)}{h}$.)

But Problem 8.7 shows that the only character of $A_5$ is the character which always takes the value 1.

**Problem 9.3.** Suppose that $f(r_1, r_2, \ldots, r_5)$ is in $F$ and $\sqrt[n]{f}$ is in $L$. Show that $\sqrt[n]{f}$ is in $F$.

**Problem 9.4.** Explain why we have proved the Theorem!