PAYNE’S THEOREM AND COEFFICIENTS OF RATIONAL POWER SERIES

DAVID E SPEYER

Let \( P \) be the set \( \{ p : p \text{ is prime} \} \cup \{ \infty \} \). For \( p \in P \), let \( |\cdot|_p \) be the \( p \)-adic absolute value on \( \mathbb{Q} \). Let \( \mathbb{C}_\infty \) be the complex numbers; for \( p \) a finite prime, let \( \mathbb{C}_p \) be the completion of the algebraic closure of \( \mathbb{Q}_p \).

Let \( f \) and \( g \in \mathbb{Q}[t_1, t_2, \ldots, t_n] \) be relatively prime polynomials with \( g(0, 0, \ldots, 0) \neq 0 \). Let

\[
    f(t_1, t_2, \ldots, t_n) = \sum_{(d_1, d_2, \ldots, d_n) \in \mathbb{N}_0^n} a(d_1, d_2, \ldots, d_n) t_1^{d_1} t_2^{d_2} \cdots t_n^{d_n}.
\]

So \( a(d_1, \ldots, d_n) \) is a rational number.

Let \( \phi : \mathbb{Z}_p^d \to \mathbb{Q} \) be a function. We define \( \phi \) to be a \textit{quasi-polynomial} if we can partition \( \mathbb{N}^d_0 \) into finitely many sets \( S_1 \cup S_2 \cup \cdots \cup S_r = \mathbb{Z}^d_0 \) where

1. Each \( S_i \) is of the form

\[
    \left\{ d \in \mathbb{Z}_0^d : \langle d, u \rangle \geq 0 \right\}
\]

where \( u^1, \ldots, u^K \) and \( e_1, \ldots, e_L \) are sequences of integer vectors and \( N_j \) is a sequence of positive integers.

2. The restriction of \( \phi \) to each \( S_k \) is a polynomial.

The point of this note is to prove the following theorem. Our key tool is a result of Sam Payne, which in turn depends on a deep Diophantine theorem of Zhang.

\textbf{Theorem.} With notation as above, the following are equivalent:

1. The polynomial \( g \) factors as \( \prod \Phi_{d_i}(t_1^{e_1} \cdots t_n^{e_n}) \) where \( \Phi_d \) is the \( d \)-th cyclotomic polynomial and \( (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n_0 \), with not all the \( e_i = 0 \).
2. The function \( (d_1, \ldots, d_n) \mapsto a(d_1, \ldots, d_n) \) is a quasi-polynomial.
3. There are constants \( C \) and \( D \) such that

\[
    |a(d_1, \ldots, d_n)|_\infty \leq C \left( \sum d_i \right)^D
\]

and, for every finite prime \( p \), there is a constant \( C_p \) such that

\[
    |a(d_1, \ldots, d_n)|_p \leq C_p.
\]

4. For every \( p \in P \), there are no zeroes of \( g(t_1, \ldots, t_n) \) in the open polydisc \( \{(u_1, \ldots, u_n) \in \mathbb{C}_p : |u_1|, |u_2|, \ldots, |u_n| < 1\} \).

\textbf{Remark:} It is easy to generalize this result to number fields \( K \) other than \( \mathbb{Q} \). The one nonobvious point is that, instead of cyclotomic polynomials, one should use irreducible factors of \( 1 - t^d \) in \( K[t] \).

\textbf{Remark:} Note that \( |\alpha|_p \) is a real number, so the inequalities in parts (3) and (4) are ordinary inequalities of real numbers.

\textbf{Remark:} When \( n = 1 \), the implication (4) \( \implies \) (1) is essentially the theorem of Kronecker: An algebraic integer all of whose Galois conjugates have norm \( \leq 1 \) is a root of unity. See [Kronecker], or see [Greiter] for a nice modern exposition.

We now prove the downward implications.

(1) \( \implies \) (2): Rewrite \( f/g \) as \( h/\prod(1-t_i^{e_i}) \), where \( h \) is some polynomial. This is the sum of finitely many rational functions whose numerator is a monomial and whose denominator is \( \prod(1-t_i^{e_i}) \). So it is enough to show that the coefficients of \( 1/\prod(1-t_i^{e_i}) \) form a quasi-polynomial. This is standard.

(2) \( \implies \) (3): This is obvious.

(3) \( \implies \) (4): Let \( g_1 g_2 \cdots g_r \) be the factorization of \( g \) into irreducibles. Suppose for the sake of contradiction that \( g_i \) has a zero in \( \{ (u_1, \ldots, u_n) \in \mathbb{C}_p : |u_1|, |u_2|, \ldots, |u_n| < 1 \} \). We will write \( Z_p(g_i) \) for the zero locus of \( g_i \) in \( \mathbb{C}_p^n \), and write \( U_p \) for the open polydisc. Then \( Z_p(g_i) \cap U_p \) is Zariski dense.
in \( Z_p(g_i) \). In particular, since \( f \) and \( g \) are relatively prime, there is a point \((x_1, \ldots, x_n)\) which is in \( Z_p(g_i) \cap U_p \) but not in \( Z_p(f) \). But then \( f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \sum a(d_1, \ldots, d_n)x_1^{d_1} \cdots x_n^{d_n} \). We know \( g(x_1, \ldots, x_n) \) is 0 and the second factor is convergent by the hypothesis (3), so \( f(x_1, \ldots, x_n) \) is zero after all, a contradiction.

The rest of this note is taken up with proving the reverse implication. For every \( p \in P \), we define a subset \( A_p(g) \) of \( \mathbb{R}^n \) as follows: \( A_p(g) \) is the closure of the image of \( Z_p(g) \cap (\mathbb{C}_p^n) \) under the map \( a \mapsto \log(|a|_p) \). For \( p = \infty \), the image is already closed and is called the amoeba of \( g \); for \( p < \infty \) this is more commonly called the tropicalization of \( g \). The archimedean and non-archimedean communities use opposite sign conventions; we follow the non-archimedean tradition. We define \( \mathcal{A}_h(g) \) to be \( \bigcup_{p \in S} A_p(g) \), which (following Payne) we term the adelic amoeba of \( g \).

We will be using the following result of Sam Payne. This is [Payne, Proposition 4.2]; the proof relies on a Diophantine result of Zhang [Zhang], see also [BZ].

**Payne’s Theorem.** Let \( h \) be a polynomial in \( \mathbb{Q}[t_1, \ldots, t_n] \). Let \((v_1, \ldots, v_n) \in \mathbb{R}^n \), with not all \( v_i \) equal to 0 and suppose that \( \mathbb{R}_{>0} : v \cap \mathcal{A}_h(h) \) is empty. Then each irreducible geometric component of \( h = 0 \) is a translate of a subtorus by a torsion point.

**Remark:** This differs from an exact quotation of Payne’s result in two ways — Payne works over a general number field, rather than \( \mathbb{Q} \), and Payne implicitly reduces to the case that \( h \) is geometrically irreducible without saying so explicitly.

Note that, in equations, a translate of a subtorus by a torsion point looks like \( 1 - \zeta t_1^{e_1} \cdots t_n^{e_n} \) where \((e_1, \ldots, e_n)\) is an integer exponent and \( \zeta \) is a root of unity.

We now conclude our proof, showing that (4) implies (1). Note that (4) can be restated as

\[
\mathcal{A}_h(g) \cap \mathbb{R}_{>0}^n = \emptyset.
\]

So, whenever \((v_1, v_2, \ldots, v_n)\) is in \( \mathbb{R}_{>0}^n \), the hypothesis of Payne’s theorem holds. So \( g(t_1, \ldots, t_n) = \prod(1 - \zeta t_1^{e_1} \cdots t_n^{e_n}) \). Since \( g \) has rational coefficients, we can group these binomials together into cyclotomic polynomials. If \((e_1, \ldots, e_n)\) do not all have the same sign, then the binomial \( 1 - \zeta t_1^{e_1} \cdots t_n^{e_n} \) has roots in the open polydisc \( U_p \) (for every \( p \), in fact), contradicting our assumption (4). So all the components of \( e^j \) have the same sign and, without loss of generality, that sign is nonnegative. This concludes our proof.

The motivation for this note was the following question of Alex Fink [Fink].

Let \( A \) be a subset of \( \mathbb{Z}_{\geq 0}^{n+1} \) and suppose that \( \sum_{d \in A} t_1^{d_1} \cdots t_n^{d_n} \in \mathbb{Z}[[t_1, \ldots, t_n]] \) is a rational function. What can be said about the structure of \( A \)?

(I have taken the liberty of renaming some of Fink’s variables.)

Let \( \chi_A : \mathbb{Z}_{\geq 0}^n \to \{0, 1\} \) be the characteristic function of \( A \). The function \( \chi_A \) obeys (3), with \( C = C_p = 1 \) and \( D = 0 \). Our theorem states that \( \chi_A \) is a quasi-polynomial. Write \( \mathbb{Z}_{\geq 0}^n \) as \( S_1 \cup \cdots \cup S_r \), as in the definition of quasi-polynomials, with \( \chi_A|S_k = \chi_A \) equal to the polynomial \( \phi_k \). Since \( \phi_k \) only assumes the values 0 and 1 on \( S_k \), we see that \( \phi_k \) must be constant; equal to either 0 or 1. Let \( S_{k_1}, \ldots, S_{k_r} \) be the sets where \( \phi_k = 1 \). So \( A = \bigcup S_{k_i} \), where each \( k_i \) is given by finitely many integer inequalities and congruences.

**References**


