MATH 451, ADVANCED CALCULUS I, Section 2

Fall Term, 2005
B.A. Taylor

ASSIGNMENTS

Text: Kenneth A. Ross, *Elementary Analysis: The Theory of Calculus.*

Chapter 1.

1. Wednesday, September 7, Read the Appendix on set notation, and §1, §2.
Individual homework problems: 1.1, 1.11, 2.3
Team homework problem 1.12

2. Friday, September 9, Read §3,§4
Individual homework problem: 3.1, 3.5, 3.6, 3.7
Team homework problem: 3.8

3. Monday, September 12, Read §3,§4
Individual homework problems: 4.1, 4.3, 4.6 , 4.9
Team homework problem: 4.5, 4.14

4. Wednesday, September 14, Read §4,§5
Individual homework problems: 4.8, 4.9, 4.13, 5.1
Team homework problems: 4.15, 5.6

Hand in Team homework problems from Assignments 1,2 on Wednesday, September 14.

Hand in individual homeworks from Assignments 1, 2, 3 on Friday, September 16.

The Team problems from Assignments 3,4 are due on Wednesday, September 21.

Chapter 2.

5. Friday, September 16 and Monday, September 19, Read §7,§8
Individual homework problems: Study, but do not hand in, 7.1, 7.2, 7.3, 7.5, 8.4, Also do, to hand in, 8.2 (b),(e), 8.5, 8.10
Team problems 8.7(a),(b), 8.8(c), and the following:

Exercise 1. Let S be a set of real numbers that has the property, "If $a \in S$, then $[a, \infty) \subset S$." Prove that S has one of the following four forms:

1. $S = \emptyset$.
2. $S = \mathbb{R}^n$.
3. $S = [b, \infty)$ *for some* $b \in \mathbb{R}$.
4. $S = (b, \infty)$ *for some* $b \in \mathbb{R}$.

(Hint: You may want to use the completeness axiom in the proof of the exercise.)

The individual problems from Assignments 4,5 are due on Friday, September 23.

6. Wednesday, September 21 and Friday, September 23. Read §9
Individual homework problems: Study but you need not hand in 9.6, 9.7, 9.8, 9.9. Hand in 9.1, 9.4, 9.5, 9.6
Team homework problems: 9.7, 9.12 and the following.
Exercise 2. Show that $s_n = \left(1 + \frac{1}{n}\right)^n$ *is a bounded increasing sequence and give an explicit upper bound for the sequence (is 10 an upper bound?). You may not use any properties or knowledge of the number e in your proof.*
(Hint: Write out the expansion of expression defining s_n using the binomial theorem.)

The Team problems from Assignments 5,6 are due on Wednesday, September 28.

7. Monday, September 26, Read §10
Individual homework problems: Study, but do not hand in, 10.1, Hand in 10.2, 10.4, 10.12
Team homework problems: 10.6, 10.7; Optional (hard) extra credit problem: 10.11

Exercise 3. Give the definition, analogous to Definition 10.8, of the statement, $\{s_n\}$ is not a Cauchy sequence if ....

The individual problems from Assignments 6,7 are due on Friday, September 30 (Exam Day).

8. Wednesday, September 28, Read §11
Individual homework problems: 11.3, 11.5
Team homework problems: none

The Team problems from Assignments 7,8 are due on Wednesday, October 5.

Friday, September 30, MIDTERM EXAM I

9. Monday, October 3, Read §12
Individual homework problems:Study, but do not hand in, the examples in 12.3; hand in 12.4, 12.8, 12.12
Team homework problem: 12.13

The individual problems from Assignments 8,9 are due on Friday, October 7.

10. Wednesday, 5 and Friday, October 7, Read §13
Individual homework problems: 13.1, 13.5, 13.9
Team homework problems 13.3, 13.13

The Team problems from Assignments 9,10 are due on Wednesday, October 12.

The individual problems from Assignment 10 are due on Friday, October 14.

11. Monday, October 10 and Wednesday, October 12. Read §14.
Individual homework problems: Study, but do not hand in, 14.1, 14.2, 14.3, 14.4. Hand in 14.5, 14.8, 14.9.
Team homework problems: 14.10, 14.12 and the following:
Exercise 4. Let $\{a_n\}$ *be a decreasing sequence of nonnegative numbers.*

(a) *Prove that* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=1}^{\infty} 2^n a_{2^n}$ *converges.*

(b) *Apply this test to prove or disprove:* $\sum_{n=1}^{\infty} \frac{1}{n}$ *converges.*

(c) *Apply this test to decide for which numbers* $p > 0$ *it is true that* $\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$ *converges.*

12. Friday, October 14, Browse §15,16, especially Theorems 16.3, 16.5 and Example 4. (Skip §15)
Individual homework problem: 16.8

The Team problems from Assignments 11,12 are due on Wednesday, October 12.

The individual problems from Assignments 11,12 are due on Friday, October 21.

Chapter 3.

13. Wednesday, October 19, Read §17
Individual problems 17.1, 17.3, 17.9, 17.10
Team problems 17.13, 17.14 and
Exercise 5. Give an example of a bounded continuous function f on $(0,1)$ and a Cauchy sequence $\{x_n\}$ in $(0,1)$ such that $f(x_n)$ is not a Cauchy sequence.

14. Friday, October 21, Read §18
Individual problems 18.3, 18.5, 18.7, 18.12
Team problems 18.2 and
Exercise 6. Let f be a real-valued function defined on a closed interval $[a,b] \subset \mathbb{R}$. The **graph of f** *is the subset of $\mathbb{R}^2$ defined by $G(f) := \{(x, f(x)) : a \leq x \leq b\}$. Show that f is continuous on $[a,b]$ if and only if $G(f)$ is a compact subset of $\mathbb{R}^2$.*

The team problems from Assignment 13 are due on Wednesday, October 26.

The individual problems from Assignments 13 and 14 are due on Friday, October 28.

15. Wednesday, October 26, Read §19
Individual problems 19.1, 19.2, 19.3, 19.5
Team problem 19.9

The team problems from Assignment 14 and 15 are due on Wednesday, November 2.

16. Friday, October 28, Read §20 (lightly)
Individual problems 20.1, 20.4, 20.11, 20.16

17. Monday, October 31, Read §21
Individual problems 21.8, 21.10, 21.11
Team problems 21.2, 21.4, 21.9

The individual problems from Assignments 15 through 17 are due on Friday, November 4.

Friday, November 4, MIDTERM EXAM II

Will cover material through Chapter 3.

18. Monday, November 7, Read §§23, 24
Individual problems 23.1, 23.2, 24.1,24.2, 24.3, 24.4
Team problems 24.17 and Exercise 7.

Exercise 7. *Write the precise mathematical description of the negation of the statement, "the sequence of real-valued functions f_n converges uniformly to f on S."*

19. Wednesday, November 9, Read §§23, 24 Individual problems 23.3, 24.10, 24.11, 24.14
Team problem 25.11

Hand in individual problems 23.2 and 24.2 from Assignment 18 on Friday, November 11.

20. Friday, November 11, Read §§25,26 Individual problems 25.6. 25.7, 25.9, 25.10, 25.12, 25.13
Team problems: Exercise 8.

Exercise 8. *Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ that converges pointwise to a continuous function f. Suppose also that if $\{x_n\}$ is a sequence of points in $[a, b]$ that converges to x, then $\lim_{n\to\infty} f_n(x_n) = f(x)$. Prove that the sequence f_n converges uniformly to f on $[a, b]$. (Compare with Exercise 24.17)*

21. Monday, November 14, Read §§25,26 Individual problems 26.2, 26.3, 26.7, 26.8
Team problems: Exercise 9. Exercise 10 is optional for 5 points of extra credit.

Exercise 9. **(a)** *Give an example of a pointwise convergent sequence of integrable functions on $[0, 1]$ such that the limit function is not integrable. Hint: remember the function $f(x) = 0$ if x is irrational, $f(x) = 1$ if x is rational.*

(b) *Give an example of a pointwise convergent sequence of integrable functions on $[0, 1]$ such that the limit function f is integrable, but $\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.*

Exercise 10. *In Chapter 3, the n-th root test (14.9) for convergence of infinite series was given. This problem asks, "is there a $\sqrt{n}$-th root test. Namely, for an infinite series $\sum_{n=1}^{\infty} a_n$, let $\alpha = \limsup_{n\to\infty} |a_n|^{1/\sqrt{n}}$. Then is it true that*

(i) *if $\alpha < 1$, the series converges absolutely?*

(ii) *if $\alpha > 1$, the series diverges?*

(iii) *if $\alpha = 1$, the series may either converge or diverge?*

Hint: Remember that the n-th root test is proved by comparing $\sum_{n=1}^{\infty} a_n$ with the geometric series $\sum_{n=1}^{\infty} (\alpha \pm \epsilon)^n$ whose convergence properties we know. You might try comparing with the series $\sum_{n=1}^{\infty} \alpha^{\sqrt{n}}$. When does it converge?

The team problems from Assignment 18, 19, and 20 are due on Wednesday, November 16.

Hand in individual problems 24.9, 25.6, 25.12, 26.2, and 26.5 from Assignments 19, 20, 21 on Friday, November 18.

22. Wednesday, November 16, Read §28 (Definition and properties of the derivative.)
Individual problems: Do, but do not hand in, 28.1, 28.2, Hand in 28.3, 28.4, 28.6, 28.9
Team problems: 28.5 and Exercise 11.

Exercise 11. *Suppose f is differentiable at $x = a$.*

(a) *Prove that* $\lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$. *Explain the geometric meaning of the limit in terms of secant lines to the graph of f, including an illustrative sketch.*

(b) *Explain why it is not necessarily true that* $\lim_{h \neq k \to 0} \frac{f(a+h) - f(a+k)}{h-k}$. *You can do this by sketching a graph of a function for which the limit would fail to exist, or by giving a specific example. What's different geometrically about the secant lines in this case and in case (a)?*

(c) *Show that if the derivative of f exists and is continuous in an open interval containing a, then the limit in part (b) does exist and is equal to $f'(a)$. Explain the geometric meaning. (This requires the use of the mean value theorem from §29.)*

23. Friday, November 18, Read §29 (Mean Value Theorem and related topics.)
Individual problems: Do, but do not hand in, 29.1, 29.2. Hand in 29.3, 29.5, 29.10, 29.11, 29.13.
Team problems: 29.16 and Exercise 12.

Exercise 12. *Prove that if $\phi(x)$ is a differentiable function on $\mathbb{R}$ such that $|\phi'(x)| \leq c < 1$, then*

(i) *The equation $x = \phi(x)$ can have at most one solution.*

(ii) *For any value of $x_0 \in \mathbb{R}$, the sequence defined recursively by $x_{n+1} = \phi(x_n)$ converges to a number that is a solution of the equation $x = \phi(x)$, hence, by part (i) the unique solution of the equation.*

Draw a graph of x and $\phi(x)$ on the same set of axes, select an x_0, and give a graphical illustration of the points x_n that are generated by this **fixed point iteration.**

24. Monday, November 21 and Wednesday, November 23, Read §31 (skip §30). Approximation of functions by Taylor's Series.
Individual problems: Do, but do not hand in, 31.1
Team problems: 31.4, 31.5, 31.6

The team problems from Assignments 21 and 22 are due on Wednesday, November 23.

The individual problems from Assignments 22 and 23 are due the Monday after Thanksgiving vacation.

The team problems from Assignments 23 and 24 are due on Wednesday, November 30.

25. Monday, November 28
Finish the worksheets on the exponential and logarithm functions (there may be questions from these worksheets on the final exam).
Individual problems: Finish the worksheet on sin and cosine functions and hand it on Wednesday, November 30.

26. Wednesday, November 30 and Friday, December 2. Read SS32,33 on the definition of the integral and its properties.
Individual problems: 32.1, 32.8, 33.3, 33.5, 33.9
Team problems: 32.3, 32.6, 33.4, 33.7, 33.10

27. Monday, December 5, Read §34 on the Fundamental Theorem of Calculus.
Individual problems: 34.2, 34.3, 34.5, 34.6, 34.8
Team problems: 34.10

28. Wednesday, December 7, Read §36 about Improper integrals (skip §35).
Individual problems: 36.3,36.4, 36.6, 36.7

The individual problems from Assignments 26 and 27 are due on Friday, December 9.

The team problems from Assignment 26 are due on Wednesday, December 7.

FINAL EXAM: Tuesday, December 20, from 4-6 pm, 3088 East Hall