

MATH 451 — FIRST MIDTERM EXAM

Solutions.

Instructor: B. A. Taylor, Section 2

Fall 2005

1. (30 pts.) Let $\{s_n\}$ be the sequence of real numbers, $s_n = \frac{1 - (-1)^n n}{n}$, $n = 1, 2, 3, \dots$

(a) Write the first 6 terms of the sequence.

$$\frac{2}{1}, \frac{-1}{2}, \frac{4}{3}, \frac{-3}{4}, \frac{6}{5}, \frac{-5}{6}$$

(b) Give the value of each of the following quantities if it exists. Otherwise, write “does not exist” (“DNE” will do).

(i) $\max\{s_n : n = 1, 2, 3, \dots\}$. 2

(ii) $\min\{s_n : n = 1, 2, 3, \dots\}$. DNE

(iii) $\sup\{s_n : n = 1, 2, 3, \dots\}$. 2

(iv) $\inf\{s_n : n = 1, 2, 3, \dots\}$. -1

(v) $\limsup_{n \rightarrow \infty} s_n$. 1

(vi) $\liminf_n s_n$. -1

(vii) $\lim s_n$. DNE

2. (20 pts.) Prove that $(5 - \sqrt{10})^{1/3}$ is not a rational number.

Be sure to state clearly any theorems you use in your proof.

Proof. Let $x = (5 - \sqrt{10})^{1/3}$. Then $x^3 = 5 - \sqrt{10}$ so $x^3 - 5 = -\sqrt{10}$. Squaring both sides of this equation then gives us that x is a root of the polynomial equation $x^6 - 10x^3 + 15 = 0$. This is an equation with integer coefficients.

Recall the Rational Zeros Theorem about solutions of polynomial equations.

Theorem. Suppose that a_0, a_1, \dots, a_n are integers and that r is a rational number satisfying the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

where $n \geq 1$, $a_n \neq 0$, and $a_0 \neq 0$. Write $r = p/q$ where p, q are integers having no common factors and $q \neq 0$. Then q divides a_n and p divides a_0 .

From the theorem, we see that the only possible rational roots of the polynomial equation with integer coefficients, $x^6 - 10x^3 + 15 = 0$, have $r = p/q$ with $p = \pm 1, \pm 3, \pm 5, \pm 15$ and $q = 1$. The number x we are considering is a root of the equation, is positive, and between 1 and 2 in magnitude. If x were rational, it would have to be equal to one of these values of $r = p/q$, and it is not. Therefore, x cannot be rational. \square

3. (30 pts) Let $s_n, n = 1, 2, 3, \dots$ be a sequence of real numbers and s a real number.

(a) Define: $\lim s_n = s$.

$\lim s_n = s$ if and only if, for each $\epsilon > 0$, there is a number N such that $n > N$ implies $|s_n - s| < \epsilon$.

(b) Define: $\limsup s_n$.

$$\limsup s_n = \lim_{n \rightarrow \infty} \sup \{s_k : k \geq n\}.$$

(c) Using your definition from part (a), prove that $\lim \frac{2n^3 + n}{n^3 - 3} = 2$.

Let $\epsilon > 0$. Let $N = \max \left\{ 6, \frac{2}{\sqrt{\epsilon}} \right\}$. Note that if $n \geq 6$, then

$$\left| \frac{2n^3 + n}{n^3 - 3} - 2 \right| = \left| \frac{n + 6}{n^3 - 3} \right| \leq \frac{n + 6}{n^3 - 3} \leq \frac{n + n}{n^3 - n^3/2} = \frac{4}{n^2}.$$

Consequently, if $n > N$, then

$$\left| \frac{2n^3 + n}{n^3 - 3} - 2 \right| \leq \frac{4}{n^2} < 4 \left(\frac{\sqrt{\epsilon}}{2} \right)^2 = \epsilon.$$

4. (20 pts.) (a) Prove that a bounded, nondecreasing sequence of real numbers $\{s_n\}$ converges.

Proof. Let $s = \sup E$, where $E = \{s_n : n = 1, 2, 3, \dots\}$. Since E is given to have an upper bound, the number s exists by the completeness axiom. We claim that $\lim_{n \rightarrow \infty} s_n = s$. To prove this, let $\epsilon > 0$. Then $s - \epsilon < \sup E$, so $s - \epsilon$ is not an upper bound for E . Consequently, there exists a natural number N such that $s_N > s - \epsilon$. Then for all $n > N$, we have

$$s - \epsilon < s_N \leq s_n \leq s$$

where the second inequality is because the sequence is nonincreasing and the third because s is an upper bound for all the terms of the sequence. Consequently, $|s_n - s| < \epsilon$ if $n > N$, so the sequence converges to s .

(b) Let E be a nonempty, bounded subset of the real numbers.

Let A be the statement: $\sup E$ is not an element of E .

Let B be the statement: E has infinitely many elements.

The statement “A implies B” written in English, is: *If $\sup E$ is not an element of E , then E has infinitely many elements.*

(i) Write the English version of the contrapositive of “A implies B”.

If E has only finitely many elements, then $\sup E \in E$.

(ii) Write the English version of the converse of “A implies B”.

If E has infinitely many elements, then $\sup E$ belongs to E .

(iii) Which of the statements “ A implies B ”, its contrapositive, and its converse are true. Why or why not?

“ A implies B ” and the contrapositive, “not B implies not A ” are both true and they are equivalent statements. Both statements are true because for any finite set of numbers, at least one of them is greater than or equal to all the others. Hence, E has a maximum element which is then also equal to the supremum.

The converse, “ B implies A ”, is false. The supremum of the infinite set $\{-1/n : n = 1, 2, 3, \dots\}$ is 0 which is not an element of the set.