Math 605, Fall Term, 2005

Homework #1. To be handed in on Monday, September 19.

In these problems, Ω denotes a domain in $\mathbb{C}^n$. Use the first definition of "analytic function" given in class, namely $f \in C^1(\Omega)$ and $\overline{\partial} f = 0$.

1. Prove that $A(\Omega)$ is a closed subspace of both $C(\Omega)$ and $\mathbb{C}^\infty(\Omega)$. Explain what this implies about a sequence of functions $f_j \in A(\Omega)$ that converges uniformly on compact subsets of Ω.

2. Prove Montel's Theorem: *A locally bounded set of functions in $A(\Omega)$ is relatively compact.* (Remark: If you don't know about equicontinuous families of continuous functions, ask Taylor.)

3. (a) Show that if f is analytic on a neighborhood of the torus $\{|z_1| = r_1, \ldots, |z_n| = r_n\}$, then f has a multiple Laurent expansion $\sum_{\alpha \in Z^n} a_\alpha z^\alpha$ that converges absolutely and uniformly to f on a neighborhood of the torus.

(b) Prove the estimate for the coefficients a_α that is analagous to Cauchy's estimate for the coefficients of the power series expansion of an analytic function.

4. Suppose f is analytic on the Reinhardt triangle, $T = \{(z_1, z_2) : 0 < |z_2| < |z_1| < 1\}$.

(a) Show that f has an expansion of the form

$$f(z_1, z_2) = \sum_{-\infty < j < +\infty} a_j(z_2) z_1^j$$

that converges to f in $A(T)$ and where each coefficient function $a_j(z_2)$ is analytic for $0 < |z_2| < 1$.

(b) Show that if f is uniformly bounded by M, then

(i): $|a_j(z_2)| \leq M$ for $j \geq 0$; and

(ii): $|a_j(z_2)| \leq M|z_2|^{|j|}$ for $j < 0$ and $|z_2| < 1$.

(c) Show that if f extends to be analytic at the point $(0, 0)$, then f extends (uniquely) to a function analytic on the polydisk $\{|z_1| < 1, |z_2| < 1\}$.

(d) (optional) What can you say if f extends to be a C^k function at (0,0) for some k = 0, 1, 2, ...? What about $k = \infty$?