Math 605, Fall Term, 2005

Homework #2. To be handed in on Monday, October 3.

1. Let u be a locally bounded function on a set $E \subset \mathbb{R}^n$ that takes values in $[-\infty, +\infty)$. The *upper semicontinuous regularization of* u is the function u^* defined by

$$u^*(x) = \limsup_{y \to x, y \in E} u(y).$$

Prove that u^* is uppersemincontinuous on E.

2. Recall that a set $\Omega \subset \mathbb{R}^n$ is convex if the line segment from x to y, $[x, y] := \{(1-t)x + ty \ : \ 0 \le t \le 1\} \subset \Omega$ whenever $x, y \in \Omega$. A function φ is said to be convex on Ω if and only if its graph lies below any chord; i.e.

$$\varphi((1-t)x + ty) \le (1-t)\varphi(x) + t\varphi(y)$$

for $x, y \in \Omega$ and $0 \le t \le 1$.

(a). Prove that a convex function φ on an open convex set Ω is Lipshitz continuous of order 1 on each compact subset of Ω; that is, for each compact set $K \subset \Omega$, there is a constant M such that $|\varphi(x) - \varphi(y)| \le M|x - y|$ for all $x, y \in K$. *Hint: Look first at the one variable case and find an explicit bound for the difference quotients of* φ.

(b) Prove that a C^2 function φ on an open set is convex in a neighborhood of each point of the set if and only if the Hessian of φ, $\left[\dfrac{\partial^2 \varphi}{\partial x_j \partial x_k}\right]$, is a positive (semi)definite matrix. That is, for all $\lambda \in \mathbb{R}^n$ and $x \in \Omega$,

$$\sum_{j,k=1}^{n} \lambda_j \lambda_k \frac{\partial^2 \varphi(x)}{\partial x_j \partial x_k} \ge 0.$$

3. **(a)** Show that a function u that is independent of $\operatorname{Im} z$ is plurisubharmonic if and only if it is a convex function of $\operatorname{Re} z$ that is nondecreasing in each variable. In case u is smooth, what is the relationship between the real Hessian of the convex function and the complex Hessian $\left[\dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right]$ of u?

(b) Show that a function u that is a function of $(|z_1|, \ldots, |z_n|)$ is plurisubharmonic if and only if it is a convex function of $(\log |z_1|, \ldots, \log |z_n|)$ that is increasing in each variable separately. That is, there is a convex function φ that is nondecreasing in each variable and satisfies

$u(z) = \varphi(\log|z_1|, \ldots, \log|z_n|)$. In case u is smooth, give the relationship between the complex Hessian of u and that of φ in the smooth case.

4. Let $p_\nu(z)$ be a sequence of homogeneous polynomials in $z = (z_1, \ldots, z_n)$ with p_ν of degree ν, $\nu = 0, 1, 2, \ldots$. Suppose also that: such thate a formal power series of . Suppose that:
(a) the power series $\sum_{\nu=0}^{\infty} p_\nu(z)$ converges absolutely to an analytic function $f(z)$ on a neighborhood of the origin; and
(b) For each fixed $z \neq 0$, the function of one complex variable, $\zeta \to f(\zeta z)$, which is analytic for ζ near the origin actually is an entire function. i.e. has an analytic continuation to all of $\mathbb{C}$.

Prove that the formal power series actually converges absolutely and uniformly on compact subsets of $\mathbb{C}^n$ to an entire function.

Hint: The homogeneous polynomials satisfy an estimate of the form $|p_\nu(z)| \leq C_\nu |z|^\nu$. *What do the hypotheses and conclusion tell us about the size of the constants* C_ν*?*