The shape of a random affine Weyl group element

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A Coxeter group \((W, S)\) is a group generated by a set 
\(S = \{s_1, s_2, \ldots, s_r\}\) of simple generators which are involutions 
satisfying relations of the form 
\[(s_is_j)^{m_{ij}} = 1\]

**Definition**

A word \(i_1i_2 \cdots i_\ell\) is a reduced word if \(\ell\) is minimal amongst 
expressions \(w = s_{i_1}s_{i_2} \cdots s_{i_\ell}\) for \(w\).
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Example (Symmetric group $S_3$)

$S_3$ is generated by involutions $s_1, s_2$ with the relation

$$s_1 s_2 s_1 = s_2 s_1 s_2$$

No infinite reduced words.
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Example (Affine symmetric group $\tilde{S}_3$)

$\tilde{S}_3$ is generated by involutions $s_0, s_1, s_2$ with relations

\[ s_1 s_2 s_1 = s_2 s_1 s_2 \quad s_0 s_1 s_0 = s_1 s_0 s_1 \quad s_2 s_0 s_2 = s_0 s_2 s_0 \]

$012012012012 \cdots$ is an infinite reduced word
Basic question

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What does a random infinite reduced word look like?
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1. Classification (up to braid equivalence), and the limit weak order for infinite reduced words in affine Weyl groups (studied with P. Pylyavskyy, also work of Ito, Cellini-Papi).

2. Infinite reduced words as geodesics in Coxeter (and Davis) complexes and relation to Tits metric on the visual boundary (studied with A. Thomas).
We will restrict ourselves to the case that $W$ is an affine Weyl group.

An affine Weyl group is a group generated by affine reflections acting cocompactly on a Euclidean space.
The $\tilde{A}_2$ arrangement

The affine symmetric group $\tilde{S}_3$ acts simply-transitively on the alcoves of this arrangement.
$A_1 \times A_1$ arrangement
The Weyl chambers are formed by the hyperplanes passing through the origin. Here there are six Weyl chambers, in bijection with the finite Weyl group $S_3$ (generated by reflections in these three hyperplanes).
The above walk corresponds to the infinite reduced word 0120210201⋯.

REDUCED = no hyperplane crossed more than once
Reduced random walk

Fix an affine Weyl group $W$.

The **reduced random walk** $X = (X_0, X_1, \ldots)$ is a sequence of alcoves in the affine Coxeter arrangement of $W$, where each step is chosen uniformly at random amongst choices which keep the walk reduced.
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Easy Facts:

1. These walks can never “get stuck”.
Fix an affine Weyl group $W$.

The reduced random walk $X = (X_0, X_1, \ldots)$ is a sequence of alcoves in the affine Coxeter arrangement of $W$, where each step is chosen uniformly at random amongst choices which keep the walk reduced.

Easy Facts:

1. These walks can never “get stuck”.
2. This process is a transient Markov chain.
Fix an affine Weyl group $W$. Let $X = (X_0, X_1, \ldots)$ be the reduced random walk.

**Theorem (L.)**

There exists a unit vector $\psi \in V$ such that almost surely

$$\lim_{k \to \infty} \nu(X_k) \in W \cdot \psi$$

where $\nu(X_k)$ denotes the unit vector pointing towards the center of the alcove $X_k$.

In other words, there is a finite collection $\{W \cdot \psi\}$ such that with probability one, the reduced random walk asymptotically approaches one of these directions.
The asymptotic directions for $\tilde{S}_3$. 
Define a Markov chain on the finite Weyl group $W_{\text{fin}}$ with transitions of probability $1/r$ (with $r = \dim V + 1$) given by either

$$w \rightarrow s_i w \quad \text{if} \quad \ell(s_i w) < \ell(w)$$

or

$$w \rightarrow r_\theta w \quad \text{if} \quad \ell(r_\theta w) > \ell(w)$$

Here $r_\theta$ is the longest reflection in $W_{\text{fin}}$, and extra transitions from $w$ to $w$ are added to make this a Markov chain.
The Markov chain for $S_3$

All transitions have probability $1/3$. Add self-loops to make this a Markov chain.
The Markov chain on $W_{\text{fin}}$ has a unique stationary distribution $\zeta : W_{\text{fin}} \to \mathbb{R}$. We have

$$\psi = \frac{1}{Z} \sum_{w \in W_{\text{fin}} : \ell(r_{\theta}w) > \ell(w)} \zeta(w) w^{-1}(\theta^\vee).$$
Since there are only finitely many Weyl chambers, the reducedness condition implies that every reduced walk will eventually stay in some Weyl chamber $C_w$. Write

$$X \in C_w$$

for this event.

**Question**

What is $\text{Prob}(X \in C_w)$?
Can you guess $\text{Prob}(X \in C_w)$?
The answer

In four dimensions, one chamber is $96$ times more likely than the least likely chamber.
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Main Theorem 3

Theorem (L.)

\[ \text{Prob}(X \in C_w) = \zeta(w^{-1}w_0) \]

where \( w_0 \in W_{\text{fin}} \) is the longest element of \( W_{\text{fin}} \).
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Conjecture

Let \( W = \tilde{S}_n \).

1. \( \psi \) is in the same direction as \( \rho^\vee \).
2. \[
\frac{\text{Prob}(X \in C_w)}{\text{Prob}(X \in C_1)} \in \mathbb{Z}.
\]
Main Theorem 3

**Theorem (L.)**

\[ \text{Prob}(X \in C_w) = \zeta(w^{-1}w_0) \]

*where* \( w_0 \in W_{\text{fin}} \) *is the longest element of* \( W_{\text{fin}} \).

**Conjecture**

Let \( W = \tilde{S}_n \).

1. \( \psi \) is in the same direction as \( \rho^\vee \).
2. \[ \frac{\text{Prob}(X \in C_w)}{\text{Prob}(X \in C_1)} \in \mathbb{Z}. \]

Many more conjectures for a multivariate generalization of the Markov chain on \( S_n \) with L. Williams, suggesting very interesting enumeration!
$n$-cores are a special class of partitions. Here we illustrate the bijection between 3-cores and Grassmannian elements of $\tilde{S}_3$. 
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The reduced word $02120\cdots$ gives the thickened line.
Corollary

There exists a piecewise-linear curve $C_n$ such that most large random $n$-cores (grown by the “reduced” process) has a shape arbitrarily close to $C_n$. 
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The limiting shape of a random $n$-core.
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**Corollary**

*There exists a piecewise-linear curve $C_n$ such that most large random $n$-cores (grown by the “reduced” process) has a shape arbitrarily close to $C_n$.***

This might be compared with Kerov and Vershik’s work on the shape of a random partition.
Taking $n \to \infty$

As $n \to \infty$, the piecewise-linear curve $C_n$, suitably scaled, approaches (one branch of) the continuous conic

$$\sqrt{x} + \sqrt{y} = 1$$

This curve has previously appeared as the limiting shape of another random process...
Continuous time TASEP on the integer lattice:

An independent random variable (waiting time) with exponential distribution is associated to each particle. The particle can jump only if the site immediately to the right is empty.
Continuous time TASEP on the integer lattice:

Initial configuration:
Continuous time TASEP on the integer lattice:

Each configuration is associated with a piece-wise linear curve, or Young diagram.
Continuous time TASEP on the integer lattice:

Johansson showed that the “limiting shape” of continuous time TASEP with exponential waiting time is exactly the same curve

\[ \sqrt{x} + \sqrt{y} = 1 \]

So for the affine symmetric group \( W = \tilde{S}_n \), and conditioning our random walk to stay in the fundamental chamber, we obtain a periodic analogue of continuous time TASEP: particles separated by distance \( n \) are conditioned to jump simultaneously.