Total positivity in combinatorics and representation theory

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A real matrix is totally nonnegative (TNN) if every minor is nonnegative.
A real matrix is totally positive (TP) if every minor is positive.
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Example

\[ M = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \]

is not TNN.
A real matrix is **totally nonnegative (TNN)** if every minor is nonnegative.  
A real matrix is **totally positive (TP)** if every minor is positive.

**Example**

\[
M = \begin{pmatrix}
1 & 3 \\
1 & 2
\end{pmatrix}
\]

is not TNN.  
But

\[
M = \begin{pmatrix}
1 & 2 & 1 \\
2 & 5 & 3 \\
1 & 3 & 3
\end{pmatrix}
\]

is TP.
Examples of total nonnegative matrices first arose in analysis. The kernels $K(x, y) = e^{xy}$ and $K(x, y) = e^{-(x-y)^2}$ are totally nonnegative in the sense that the matrix

$$M = (K(x_i, y_j))_{i,j=1}^k$$

is totally nonnegative for every

$$x_1 < x_2 < \cdots < x_k$$
$$y_1 < y_2 < \cdots < y_k$$

We call these totally positive kernels.
Properties of totally positive matrices

Theorem (Gantmacher and Krein)

A **totally positive matrix** (or kernel) has positive and simple eigenvalues.
Properties of totally positive matrices

**Theorem (Gantmacher and Krein)**

A totally positive matrix (or kernel) has positive and simple eigenvalues.

**Theorem (Schoenberg, Gantmacher and Krein, Karlin)**

Let $K(x, y)$ be a TP kernel and $f(y)$ a function satisfying suitable integrability conditions. Then

$$g(x) = \int_{\mathbb{R}} K(x, y)f(y)dy$$

has no more sign-changes than $f(y)$. 
Total positivity in $GL_n(\mathbb{R})$

Theorem (Loewner-Whitney)

$GL_n(\mathbb{R})_{\geq 0}$ is the semigroup generated by the positive diagonal matrices and positive Chevalley generators $e_i(t), f_i(t)$ with $t \geq 0$
Total positivity in $GL_n(\mathbb{R})$

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$GL_n(\mathbb{R})_{\geq 0}$ is the semigroup generated by the positive diagonal matrices and positive Chevalley generators $e_i(t), f_i(t)$ with $t \geq 0$

$$
\text{diag}(t_1, t_2, t_3, t_4) = 
\begin{pmatrix}
    t_1 & 0 & 0 & 0 \\
    0 & t_2 & 0 & 0 \\
    0 & 0 & t_3 & 0 \\
    0 & 0 & 0 & t_4 \\
\end{pmatrix}
\quad t_1, t_2, t_3, t_4 > 0
$$
Theorem (Loewner-Whitney)

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$$e_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad f_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
A sequence $a_0, a_1, \cdots$ of real numbers is a **totally positive sequence** if the infinite Toeplitz matrix

$$
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & a_0 & a_1 & a_2 & a_3 & \cdots \\
\vdots & 0 & a_0 & a_1 & a_2 & \cdots \\
\vdots & 0 & 0 & a_0 & a_1 & \cdots \\
\vdots & 0 & 0 & 0 & a_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{pmatrix}
$$

is TNN (caution: not TP!).
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$a_i \geq 0$
Totally positive functions

A sequence $a_0, a_1, \cdots$ of real numbers is a totally positive sequence if the infinite Toeplitz matrix

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\cdots & 0 & a_0 & a_1 & a_2 & \cdots \\
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\cdots & 0 & 0 & 0 & a_0 & \cdots \\
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$$a_i \geq 0 \quad a_1^2 - a_0a_2 \geq 0$$
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$$a_i \geq 0 \quad a_1^2 - a_0 a_2 \geq 0 \quad a_1^3 + a_3 a_0^2 - 2a_0 a_1 a_2 \geq 0$$
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$$a_i \geq 0 \quad a_1^2 - a_0 a_2 \geq 0 \quad a_1^3 + a_3 a_0^2 - 2a_0 a_1 a_2 \geq 0$$

The formal power series $a(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ is then called a totally positive function. Totally positive functions form a semigroup.
Fact: If $a(t)$ is totally positive, then it is automatically a meromorphic function, holomorphic in a neighborhood of 0.
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Theorem

*Every normalized ($a_0 = a_1 = 1$) totally positive sequence has generating function of the form*

$$a(t) = e^{\gamma t} \prod_{i=1}^{\infty} \frac{1 + \alpha_i t}{1 - \beta_i t}$$

*where $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq 0$, $\gamma \geq 0$, and $\gamma + \sum_i (\alpha_i + \beta_i) = 1$. Conversely, all such sets of parameters give a normalized totally positive function.*
Fact: If \( a(t) \) is totally positive, then it is automatically a meromorphic function, holomorphic in a neighborhood of 0.

**Theorem**

Every normalized \((a_0 = a_1 = 1)\) totally positive sequence has generating function of the form

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a(t) = e^{\gamma t} \prod_{i=1}^{\infty} \frac{1 + \alpha_i t}{1 - \beta_i t}
\]

where \( \alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \beta_1 \geq \beta_2 \geq \cdots \geq 0, \gamma \geq 0, \) and \( \gamma + \sum_i (\alpha_i + \beta_i) = 1. \) Conversely, all such sets of parameters give a normalized totally positive function.

The proof of this theorem relies on deep results in complex analysis (Nevanlinna theory).
Theorem (Thoma, Kerov-Vershik)

The following sets are in bijection (homeomorphic):

- Normalized totally positive functions.
Totally positive functions parametrize many things

**Theorem (Thoma, Kerov-Vershik)**

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- Characters $\chi$ of the infinite symmetric group $S_\infty$. 
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- (Normalized) homomorphisms $\phi : \text{Sym} \to \mathbb{R}$ such that $\phi(s_\lambda) \geq 0$ for each Schur function $s_\lambda$. 
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The following sets are in bijection (homeomorphic):

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- Characters $\chi$ of the infinite symmetric group $S_\infty$.
- (Normalized) homomorphisms $\phi : \text{Sym} \to \mathbb{R}$ such that $\phi(s_\lambda) \geq 0$ for each Schur function $s_\lambda$.
- Extremal Markov chains on Young’s lattice of partitions, such that the probability of a tableau only depends on its shape.
The infinite symmetric group

Infinite symmetric group

The symmetric group $S_n$ permuting $n$ elements embeds into $S_{n+1}$ as the subgroup fixing $n+1$. The inductive limit $S_\infty$

$$S_1 \subset S_2 \subset \cdots \subset S_\infty$$

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A character $\chi$ of $S_\infty$ is a function $\chi : S_\infty \to \mathbb{C}$ that is central, positive definite, normalized, and extremal.
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Under the correspondence (Thoma)

$$\{\text{totally positive functions}\} \leftrightarrow \{\text{characters of } S_\infty\}$$

one has (Okounkov)

poles and zeroes $\{\alpha_i, \beta_i\} \leftrightarrow$ atoms of particular spectral measure
A partition $\lambda$ of $n$ is a sequence

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$$

of nonnegative integers, such that $\lambda_1 + \lambda_2 + \cdots = n$. 
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**Example**

$(4, 3, 1, 1)$ is a partition of 9.
A partition \( \lambda \) of \( n \) is a sequence

\[
\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)
\]

of nonnegative integers, such that \( \lambda_1 + \lambda_2 + \cdots = n \).

**Example**

\((4, 3, 1, 1)\) is a partition of 9.

We have

\[
\text{irreps of } S_n \iff \text{partitions of } n
\]
Sym denotes the ring of formal power series of bounded degree in the variables $x_1, x_2, \ldots$, invariant under action of $S_\infty$ on the indices. There is an isomorphism (Frobenius character)

$$
\text{Sym} = \text{Sym}_\mathbb{R} \cong \bigoplus_{n \geq 0} \text{Rep}(S_n) \otimes \mathbb{R}
$$

where

irrep labeled by $\lambda \leftrightarrow \text{Schur function } s_\lambda(x_1, x_2, \ldots)$. 

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A character \( \chi : S_\infty \to \mathbb{C} \) gives rise to a homomorphism \( \phi : \text{Sym} \to \mathbb{R} \), where

\[
\phi(s_\lambda) = \text{coefficient of } \chi_\lambda \text{ in } \chi|_{S_n}
\]
Symmetric functions

Sym denotes the ring of formal power series of bounded degree in the variables \(x_1, x_2, \ldots\), invariant under action of \(S_\infty\) on the indices. There is an isomorphism (Frobenius character)

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\[
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\]

and

poles and zeroes \(\{\alpha_i, \beta_i\} \leftrightarrow \text{what to specialize the variables } x_1, x_2, \ldots\)
Young’s graph arises from containment of partitions:
Random partitions

Markov chains $X_0, X_1 \ldots$ on this graph, such that

- $X_i$ is a partition of $i$. 
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- The probability

$$p(\lambda^{(n)}) := \mathbb{P}(X_0 = \lambda^{(0)}, X_1 = \lambda^{(1)}, \ldots, X_n = \lambda^{(n)})$$

only depends on $\lambda^{(n)}$. 
Markov chains $X_0, X_1 \ldots$ on this graph, such that

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- the probability function $p(\lambda)$ is not a nonnegative linear combination of similar probability functions

are in bijection with normalized TP-functions.
Random partitions

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  only depends on $\lambda^{(n)}$
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are in bijection with normalized TP-functions.

Under this correspondence

\[ p(\lambda) = \phi(s_\lambda) \]

and (Kerov-Vershik)

poles and zeroes $\{\alpha_i, \beta_i\} \leftrightarrow$ scaled lengths of $i$-th rows and columns
Two variations

Block-Toeplitz:

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & a_0 & a_1 & a_2 & a_3 & \cdots \\
\cdots & 0 & b_0 & b_1 & b_2 & \cdots \\
\cdots & 0 & 0 & a_0 & a_1 & \cdots \\
\cdots & 0 & 0 & 0 & b_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

Finite:

\[
\begin{pmatrix}
1 & a_0 & a_1 & a_2 & a_3 \\
0 & 1 & a_0 & a_1 & a_2 \\
0 & 0 & 1 & a_0 & a_1 \\
0 & 0 & 0 & 1 & a_0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Consider the formal loop group $GL_n(\mathbb{R}((t)))$ consisting of (invertible) $n \times n$ matrices, whose entries are formal Laurent series. To each such matrix $X(t)$ we can associate an infinite periodic (block-Toeplitz) matrix $A(X)$:

$$
\begin{pmatrix}
1 + t^2 & 2 + 5t \\
-1 - t & -4t^2
\end{pmatrix}
\sim
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & 0 & 5 & 1 & 0 & 0 & 0 & \ddots \\
\vdots & -1 & 0 & 0 & -4 & 0 & 0 & \ddots \\
\vdots & 1 & 2 & 0 & 5 & 1 & 0 & \ddots \\
\vdots & -1 & 0 & -1 & 0 & 0 & -4 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
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\end{pmatrix} \quad \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]

A matrix $X(t) \in GL_n(\mathbb{R}((t)))$ is TNN if $A(X)$ is.
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$$
\begin{pmatrix}
1 + t^2 & 2 + 5t \\
-1 - t & -4t^2
\end{pmatrix} \leadsto
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\ldots & 0 & 5 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & -1 & 0 & 0 & -4 & 0 & 0 & \ldots \\
\ldots & 1 & 2 & 0 & 5 & 1 & 0 & \ldots \\
\ldots & -1 & 0 & -1 & 0 & 0 & -4 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 
\end{pmatrix}
$$

A matrix $X(t) \in GL_n(\mathbb{R}((t)))$ is TNN if $A(X)$ is. Note that $A(X(t)Y(t)) = A(X)A(Y)$. 
So the study of totally positive functions fits into the framework of infinite-dimensional Lie groups. Lusztig previously extended total positivity to real reductive groups.
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The case where $X = X(0)$ corresponds to $GL_n(\mathbb{R})_{\geq 0}$. The case $n = 1$ corresponds to totally positive functions.

Like for totally positive functions, a matrix $X(t) \in GL_n(\mathbb{R}((t)))_{\geq 0}$ is automatically meromorphic (every matrix entry is a meromorphic function).
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Like for totally positive functions, a matrix $X(t) \in GL_n(\mathbb{R}((t)))_{\geq 0}$ is automatically meromorphomic (every matrix entry is a meromorphic function).

Which matrices play the role of poles and zeroes for $GL_n(\mathbb{R}((t)))$?
Whirls $M(\beta_i^{(1)}, \ldots, \beta_i^{(n)})$, and curls $N(\alpha_i^{(1)}, \ldots, \alpha_i^{(n)})$, depending on $n$ (positive) parameters. Let $n = 2$.

\[
M(a, b) = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 1 & a & 0 & 0 & \ldots \\
\ldots & 0 & 1 & b & 0 & \ldots \\
\ldots & 0 & 0 & 1 & a & \ldots \\
\ldots & 0 & 0 & 0 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

\[
N(a, b) = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 1 & a & ab & a^2 b & \ldots \\
\ldots & 0 & 1 & b & ab & \ldots \\
\ldots & 0 & 0 & 1 & a & \ldots \\
\ldots & 0 & 0 & 0 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]
For $n = 1$, $M(\alpha)$ and $N(\beta)$ are exactly $(1 + \alpha t)$ and $1/(1 - \beta t)$. 
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Unlike the factors $(1 + \alpha t)$ and $(1 + \beta t)$, two whirls $M(\alpha_1, \ldots, \alpha_n)$ and $M(\beta_1, \ldots, \beta_n)$ do not always commute, but satisfy a commutation relation involving a rational transformation of the parameters.
Whirls and curls

For $n = 1$, $M(\alpha)$ and $N(\beta)$ are exactly $(1 + \alpha t)$ and $1/(1 - \beta t)$.

Unlike the factors $(1 + \alpha t)$ and $(1 + \beta t)$, two whirls $M(\alpha_1, \ldots, \alpha_n)$ and $M(\beta_1, \ldots, \beta_n)$ do not always commute, but satisfy a commutation relation involving a rational transformation of the parameters.

**Theorem (L.-Pylyavskyy)**

The transformation $(\alpha, \beta) \mapsto (\alpha', \beta')$, where $M(\alpha)M(\beta) = M(\alpha')M(\beta')$ is the “birational R-matrix” for the symmetric power representation of $U_q(\hat{sl}_n)$.

The $R$-matrix $R : V \otimes W \cong W \otimes V$ interchanges factors in tensor products of representations of quantum groups.
Theorem (L.-Pylyavskyy)

Every upper triangular $X \in GL_n(\mathbb{R}((t)))$ can be factorized as

$$\prod_{i=1}^{\infty} N(\alpha_i^{(1)}, \ldots, \alpha_i^{(n)}) \exp(Y(t)) \prod_{i=-\infty}^{-1} M(\beta_i^{(1)}, \ldots, \beta_i^{(n)})$$

for suitable positive parameters, where $Y(t)$ is entire.

Furthermore, the three factors are unique.
Theorem (L.-Pylyavskyy)

**Every upper triangular** $X \in GL_n(\mathbb{R}((t)))$ **can be factorized as**

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\prod_{i=1}^{\infty} N(\alpha_i^{(1)}, \ldots, \alpha_i^{(n)}) \exp(Y(t)) \prod_{i=-\infty}^{-1} M(\beta_i^{(1)}, \ldots, \beta_i^{(n)})
\]

**for suitable positive parameters, where** $Y(t)$ **is entire.**

**Furthermore, the three factors are unique.**

There is a ring $L\text{Sym}$ with a distinguished spanning set, called **Loop symmetric functions**, such that

TNN points of loop group $\leftrightarrow$ positive homomorphisms of $L\text{Sym}$.
Infinite products of Chevalley generators

What's going on in the $\exp(Y(t))$ part?
Infinite products of Chevalley generators

What's going on in the $\exp(Y(t))$ part?

This part of $GL_n(\mathbb{R}((t)))_{\geq 0}$ contains elements of the form

$$X = e_{i_1}(t_1)e_{i_2}(t_2)e_{i_3}(t_3) \cdots$$

where $\{e_i(t) \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ are the (affine) Chevalley generators.

$n = 3$

$$e_1(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad e_2(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \quad e_0(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ at & 0 & 1 \end{pmatrix}$$
What's going on in the $\exp(Y(t))$ part?

This part of $GL_n(\mathbb{R}((t)))_{\geq 0}$ contains elements of the form

$$X = e_{i_1}(t_1)e_{i_2}(t_2)e_{i_3}(t_3)\cdots$$

where $\{e_i(t) \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ are the (affine) Chevalley generators.

Very often these can also be written as

$$X = e_{j_1}(t'_1)e_{j_2}(t'_2)e_{j_3}(t'_3)\cdots$$

This leads to a notion of braid limits inside Coxeter groups.
Example of braid limit

Take $n = 3$, and the affine symmetric group with simple generators $s_0, s_1, s_2$ satisfying

$$s_0^2 = s_1^2 = s_2^2 = 1$$
Example of braid limit

Take $n = 3$, and the affine symmetric group with simple generators $s_0, s_1, s_2$ satisfying

$$s_0^2 = s_1^2 = s_2^2 = 1$$

$$s_0s_1s_0 = s_1s_0s_1 \quad s_1s_2s_1 = s_2s_1s_2 \quad s_0s_2s_0 = s_2s_0s_2$$
Example of braid limit

Take $n = 3$, and the affine symmetric group with simple generators $s_0, s_1, s_2$ satisfying

$$s_0^2 = s_1^2 = s_2^2 = 1$$

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\[0120212012012\cdots\]
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You can’t go back!!!
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Theorem (L.-Pylyavskyy)

*Can always end up at an infinite power of a Coxeter element. For $n = 3$: $(012)^\infty, (120)^\infty, (201)^\infty, (210)^\infty, (102)^\infty, (021)^\infty*
What about finite totally nonnegative matrices of the form

\[ M = \begin{pmatrix}
1 & a_0 & a_1 & a_2 & a_3 \\
0 & 1 & a_0 & a_1 & a_2 \\
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0 & 0 & 0 & 1 & a_0 \\
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\end{pmatrix} \]
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This problem was studied by Rietsch.
Consider all finite Toeplitz matrices form with complex entries, which is an algebraic variety $X$ isomorphic to $\mathbb{C}^{n-1}$. 
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**Theorem (Ginzburg, Peterson)**

We have a canonical isomorphism

$$\mathcal{O}(X) \cong H_*(\text{Gr}_{SL_n}, \mathbb{C})$$

between the ring of functions $\mathcal{O}(X)$ and the homology of the affine Grassmannian $\text{Gr}_{SL_n} = SL_n(\mathbb{C}((t)))/SL_n(\mathbb{C}[t])$. 
The space $\text{Gr}_{SL_n}$ is an ind-scheme, with distinguished subvarieties called Schubert varieties.
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The space $\text{Gr}_{SL_n}$ is weak homotopy-equivalent to the based loop space

$$\Omega SU(n) = \{ f : S^1 \to SU(n) \mid f(1) = 1 \}$$

giving $H_*(\text{Gr}_{SL_n})$ a ring structure.
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giving $H_*(\text{Gr}_{SL_n})$ a ring structure.

3. The ring $H_*(\text{Gr}_{SL_n})$ contains a distinguished basis $\{\sigma_w\}$ called the **Schubert basis**:

$$H_*(\text{Gr}_{SL_n}) = \bigoplus_w \mathbb{C} \cdot \sigma_w$$
Schubert positivity

Theorem (Rietsch, translated via the next theorem)

$M \in X(\mathbb{R})$ is “totally positive” $\iff \sigma_w(M) > 0$ for all $w$. 
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Rietsch’s result is originally stated in terms of the quantum cohomology \( QH^*(GL_n/B) \) of the flag variety replacing \( H_*(Gr_{SL_n}) \).

Theorem (Peterson; L.-Shimozono)

\( QH^*(GL_n/B) \) and \( H_*(Gr_{SL_n}) \) (together with their Schubert bases) can be identified after localization.
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\( QH^\ast(GL_n/B) \) and \( H^\ast(\text{Gr}_{SL_n}) \) (together with their Schubert bases) can be identified after localization.

There is also a “parametrization” result which is of a flavor different to the Edrei-Thoma theorem.
Theorem (L.)

The following sets are in bijection:

- The totally nonnegative finite Toeplitz matrices.
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- Homomorphisms $\phi : \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}] \to \mathbb{R}$ such that $\phi(s_{\lambda}^{(k)}) \geq 0$ for each $k$-Schur function $s_{\lambda}^{(k)}$.

$\mathbb{Z}[h_1, h_2, \ldots, h_{n-1}] \subset \text{Sym}$ is generated by the first $n - 1$ homogeneous symmetric functions.

$s_{\lambda}^{(k)}$ is the $k$-Schur function of Lapointe-Lascoux-Morse (with $k = n - 1$, and $t = 1$) occurring in the study of Macdonald polynomials.

Theorem (L.)

There is an isomorphism $H_*(\text{Gr}_{SL_n}) \cong \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}]$ sending Schubert classes to $k$-Schur functions.
Theorem (L.)

The following sets are in bijection:

- The totally nonnegative finite Toeplitz matrices.
- Homomorphisms $\phi : \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}] \to \mathbb{R}$ such that $\phi(s^{(k)}_{\lambda}) \geq 0$ for each $k$-Schur function $s^{(k)}_{\lambda}$.
- Extremal Markov chains on the graph of $n$-cores, such that the probability of a tableau only depends on its shape.
In Young’s lattice, boxes are added one at a time. In the graph of $n$-cores, many boxes can be added at the same step.
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For $n = 3$
Why should Schubert positivity have anything to do with total positivity?
Schubert vs. total positivity via Geometric Satake

Why should Schubert positivity have anything to do with total positivity?

Write $G = SL_n(\mathbb{C})$.

Geometric Satake Correspondence (Ginzburg, Lusztig, Mirkovic-Vilonen)

Cat. of $G([[t]])$-equivariant perverse sheaves on $Gr_G \cong \text{Rep}(G^\vee)$
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**Geometric Satake Correspondence (Ginzburg, Lusztig, Mirkovic-Vilonen)**

The category of $G([[t]])$-equivariant perverse sheaves on $\text{Gr}_G \cong \text{Rep}(G^\vee)$

$$H^*(IC_\lambda) \leftrightarrow V_\lambda$$

where $IC_\lambda$ is an intersection homology sheaf, and $V_\lambda$ is a highest-weight representation.
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where $IC_\lambda$ is an intersection homology sheaf, and $V_\lambda$ is a highest-weight representation.

$MV$-cycles $\subset Gr_G \leftrightarrow$ weight vectors in irreps of $G^\vee$
Each MV-cycle $Z \subset \text{Gr}_{SL_n}$ is an effective cycle.
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So $[Z] \in H_\ast(\text{Gr}_{SL_n})$ is a nonnegative linear combination of the Schubert classes $\sigma_w \in H_\ast(\text{Gr}_{SL_n})$. 
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**Sketch proof**

$g \in X$ is Schubert positive
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$g \in X$ is Schubert positive

$\implies g$ acts positively on the “MV-cycle basis” of irreps $V_\lambda$
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$\implies g$ is totally nonnegative (Fomin-Zelevinsky).
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Sketch proof

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\[ \implies g \text{ acts positively on the “MV-cycle basis” of irreps } V_\lambda \]
\[ \implies g \text{ is totally nonnegative (Fomin-Zelevinsky)}. \]

This is consistent with Lusztig’s definition of totally nonnegative elements as those that act positively on the canonical basis, and the general philosophy that MV-cycles are a geometric canonical basis.