Math 131 Final Exam

Wednesday, May 26, 2004

1. (15 points) A proper map \( f : X \to Y \) is, by definition, a continuous map where the inverse image of a compact set is compact. Let \( X^* = X \cup \{\infty\} \) be the one-point compactification of \( X \), and let \( Y^* \) be the one-point compactification of \( Y \). For an arbitrary function \( f : X \to Y \), define \( f^* : X^* \to Y^* \) by

\[
\begin{align*}
  f^*(x) &= \begin{cases} 
  \infty & x = \infty \\
  f(x) & \text{otherwise.}
  \end{cases}
\end{align*}
\]

Show that \( f^* \) is continuous if and only if \( f \) is continuous and proper. Conclude that a proper map to a locally compact Hausdorff space is a closed map.

2. Let \( H \) be the Hawaiian earring

\[
H = \bigcup_{n \in \mathbb{N}} C_n,
\]

the union over \( n \in \mathbb{N} \) of circles \( C_n \) of radius \( r^n \) centered at \( (0, -r^n) \) (with \( r < 1 \)). Let \( W_\infty \) be the graph with one vertex \( x_0 \) and a countable infinite set of edges \( E = \{e_n : n \in \mathbb{N}\} \). Recall that this is the quotient space of an infinite disjoint union of intervals (one for each \( n \)) where all endpoints are identified. (\( W_\infty \) is also known as the wedge of an infinite number of circles.)

(a) (10 points) Show that \( W_\infty \) is normal but is not first countable and so is not a metric space.

(b) (20 points) Consider the loops \( \gamma_n \) in \( H \) (resp. \( W_\infty \)) which run clockwise around \( C_n \) (resp. along \( e_n \) from start to end). In which topologies on \( C([0, 1], H) \) (resp. \( C([0, 1], W_\infty) \)) does the sequence of paths

\[
f_n = \gamma_1 * (\gamma_2 * (\gamma_3 * \cdots * (\gamma_{n-2} * (\gamma_{n-1} * \gamma_n)) \cdots))
\]
converge? When does the sequence
\[ g_n = ((\cdots ((\gamma_1 \ast \gamma_2) \ast \gamma_3) \cdots \ast \gamma_{n-2}) \ast \gamma_{n-1}) \ast \gamma_n \]
converge? Consider the product, uniform, compact convergence, and compact open topologies.

3. Recall that the Klein bottle \( K \) is obtained by taking the quotient space of the square

with the equivalence relation where opposite edges are identified ("glued") according to the arrows above.

(a) (10 points) Find an explicit covering map from \( \mathbb{R}^2 \) to \( K \). (Hint: Think about how we showed that \( \mathbb{R}^2 \) is the universal covering space of the torus.)

(b) (10 points) Use the map you found above to find the fundamental group of the Klein bottle. Explicitly specify the multiplication and inverse.

(c) (10 points) Use the Seifert-van Kampen theorem to give another description of \( \pi_1(K) \).

4. The degree of a covering map \( p : E \to B \) at a point \( x \in B \) is the number of elements in \( p^{-1}\{x\} \); it may be a finite number or \( \infty \).

(a) (10 points) Show that the degree is a continuous map from \( B \) to \( \mathbb{N} \cup \infty \) in the discrete topology. Conclude that if \( B \) is connected the degree is constant, so we may talk about the degree of the covering space.

(b) (15 points) Find all covering spaces of degree 2 of the theta graph

\[ \Gamma = R \quad G \quad B. \]

For each one, find the corresponding set action of \( \pi_1(\Gamma, x_0) \) and (if the covering space is connected) generators for the corresponding subgroup of \( \pi_1(\Gamma, x_0) \), using the classification of covering spaces we developed.