Some extra problems

Many of the problems are “True or False” questions. (That usually makes them harder.) If true, you should prove the statement. If false, you should provide a counterexample.

Some of the later questions are quite difficult and should be considered challenge problems.

1. Let $A$ and $B$ be square matrices of the same size. True or False:


2. True or False: there is a $3 \times 3$ matrix $A$ so that $A^7 = I$, but $A, A^2, \ldots, A^6$ are all different from $I$.

3. Let $A$ be a square matrix such that the sum of the entries in each row adds up to 1. What can you say about the matrix $I - A$?

4. Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$. Suppose

   $$\text{span}\{v_1, v_2, \ldots, v_k, x\} = \text{span}\{v_1, v_2, \ldots, v_k\}.$$  

   Prove that $x \in \text{span}\{v_1, v_2, \ldots, v_k\}$.

5. Bob thinks that

   $$A = \begin{bmatrix} -3 & 1 & 2 \\ -1 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$$

   is the square $A = B^2$ of another matrix $B$. Do you believe him?

   Alice thinks that the same matrix $A$ can be written as $A = CC^T$ for some (possibly non-square) matrix $C$. Do you believe her?

6. Let $A$ be the same matrix as in the previous problem. Suppose $v_1, v_2, v_3 \in \mathbb{R}^3$ are linearly independent vectors. True or False: it is always the case that $Av_1, Av_2, Av_3$ are linearly independent.

7. Take the same matrix $A$ again. Change at most one number in $A$ to obtain a matrix $B$, and find a vector $v \in \mathbb{R}^3$, so that $Bx = v$ has no solution. 

   This is the kind of thing I find myself doing a lot before an exam or quiz...

8. Let $A$ and $B$ be any two matrices so that the transformation

   $$x \mapsto ABx$$

   is one-to-one. True or False: $x \mapsto Ax$ is one-to-one. (Also, same question but for $B$.)
9. Let $A, B, C$ be $2 \times 2$ matrices, and suppose that $A$ and $C$ are invertible. Show that the $4 \times 4$ matrix (composed of $2 \times 2$ blocks)
\[
\begin{bmatrix}
A & B \\
0 & C
\end{bmatrix}
\]
is also invertible, and write down its inverse.

10. Prove that given any five vectors $v_1, v_2, \ldots, v_5 \in \mathbb{R}^3$, one can find numbers $c_1, c_2, \ldots, c_5$ such that both
\[
c_1v_1 + c_2v_2 + \cdots + c_5v_5 = 0 \quad \text{and} \quad c_1 + c_2 + \cdots + c_5 = 0.
\]

11. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix $A$. Let $S : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation with standard matrix $A^T$. Prove or disprove: $T$ is one-to-one if and only if $S$ is onto.

12. The trace $tr(A)$ of a square $n \times n$ matrix $A$ is the sum $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$ of diagonal entries. Prove that $tr(AB) = tr(BA)$ for two $n \times n$ matrices $A$ and $B$.

13. Let $A$ be a square matrix. We know (do we?) that $A$ is invertible if and only if we can find a square matrix $B$ such that $AB = I$.

True or False: $A$ is singular if and only if we can find a square matrix $B$ such that $AB = 0$.

14. Let $a, b, c$ be numbers. Show that the determinant of the matrix
\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & a & b \\
1 & a^2 & b^2
\end{bmatrix}
\]
is $-(a - b)(b - c)(a - c)$. Generalize this to $n \times n$ matrices. Can you prove it?

15. Let $A$ be the $2 \times 2$ matrix corresponding to a rotation by $45^\circ$. Let $B$ be the $2 \times 2$ matrix corresponding to reflection in the $x$-axis. How many different matrices can you get by multiplying a bunch of $A$’s and a bunch of $B$’s in some order? For example, matrices such as
\[
ABBA, \quad \text{or} \quad AAABABA.
\]

16. Suppose that $A = -A^T$ (such a matrix is called skew-symmetric). Show that $I + A$ is invertible.

17. Let $A$ and $B$ be $2 \times 2$ matrices. Is it always possible to find numbers $a, b$ not both zero, so that $\det(aA + bB) = 0$?