Book Homework #14 Answers
Math 217 W11

6.5.12.

a) \( \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \)

b) \( \mathbf{x} = \begin{pmatrix} 1/3 \\ 14/3 \\ -5/3 \end{pmatrix} \)

6.5.16. \( \mathbf{x} = \begin{pmatrix} 2.9 \\ 9 \end{pmatrix} \)

6.5.18.

a) True. (Paragraph following def'n of least-squares solution)

b) False. (Figure 1 and the preceding discussion)

c) True. (Equation (1) and following discussion)

d) False. (This formula only applies when the columns of \( A \) are linearly independent.)

e) True. (Equation (1) and following discussion)

f) False. (“Numerical Note”)

6.5.20. Suppose that \( Ax = 0 \). Then \( A^T Ax = A^T 0 = 0 \). Since \( A^T A \) is invertible, by hypothesis, \( x = 0 \). Hence the columns of \( A \) are linearly independent.

6.5.22. \( A^T A \) has \( n \) columns. Then \( \text{rank } A^T A = n - \dim \text{Nul } A^T A = n - \dim \text{Nul } A = \text{rank } A \), where the first and last equalities are by the rank-nullity theorem and the middle one uses Exercise 19 (which is odd and has a solution in the book).

6.7.9.

a) \( \hat{p}_2(t) = 5 \)

b) \( (p_2 - \hat{p}_2)(t) = t^2 - 5 \) completes an orthogonal basis. The multiple \( q(t) = \frac{1}{4}(t^2 - 5) \) is correctly normalized.

6.7.10.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>( p_0 )</th>
<th>( p_1 )</th>
<th>( q )</th>
<th>( p(t) = t^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>[ 1 ]</td>
<td>[ -1 ]</td>
<td>[ 1 ]</td>
<td>[ -27 ]</td>
</tr>
<tr>
<td></td>
<td>[ 1 ]</td>
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<td>[ 1 ]</td>
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<tr>
<td></td>
<td>[ 1 ]</td>
<td>[ 3 ]</td>
<td>[ 1 ]</td>
<td>[ 1 ]</td>
</tr>
</tbody>
</table>

\( \hat{p}(t) = \frac{0}{7} p_0(t) + \frac{164}{20} p_1(t) + \frac{0}{7} q(t) = \frac{41}{5} t \)

6.7.14. We check the defining properties of an inner product in turn.

1.

\( \langle u, v \rangle = T(u) \cdot T(v) \quad \text{(definition of } \langle \cdot, \cdot \rangle) \)

\( = T(v) \cdot T(u) \quad \text{(commutativity of dot product) } \)

\( = \langle v, u \rangle \quad \text{(definition of } \langle \cdot, \cdot \rangle) \)
2. 
\[ \langle u + v, w \rangle = T(u + v) \cdot T(w) \quad \text{(definition of} \, \langle \cdot, \cdot \rangle) \]
\[ = (T(u) + T(v)) \cdot T(w) \quad \text{(linearity of} \, T) \]
\[ = T(u) \cdot T(w) + T(v) \cdot T(w) \quad \text{(dot product distributes over addition)} \]
\[ = \langle u, w \rangle + \langle v, w \rangle \quad \text{(definition of} \, \langle \cdot, \cdot \rangle) \]

3. 
\[ \langle cu, v \rangle = T(cu) \cdot T(v) \quad \text{(definition of} \, \langle \cdot, \cdot \rangle) \]
\[ = (cT(u)) \cdot T(v) \quad \text{(linearity of} \, T) \]
\[ = c(T(u) \cdot T(v)) \quad \text{(bilinearity of dot product)} \]
\[ = c\langle u, v \rangle \quad \text{(definition of} \, \langle \cdot, \cdot \rangle) \]

4. For each \( u \), we have \( \langle u, u \rangle = T(u) \cdot T(u) \geq 0 \) (Theorem 1d). If \( u = 0 \), then by linearity \( T(u) = 0 \), and thus \( \langle 0, 0 \rangle = 0 \cdot 0 = 0 \). Finally, if \( \langle u, u \rangle = 0 \), then \( T(u) \cdot T(u) = 0 \), so \( T(u) = 0 \) by Theorem 1d. Since \( T \) is an isomorphism, this implies \( u = 0 \).

(Note that this part, and only this part, fails if we merely assume that \( T \) is linear.)

6.7.16. 
\[ \| u - v \|^2 = \langle u - v, u - v \rangle \]
\[ = \langle u, u - v \rangle - \langle v, u - v \rangle \]
\[ = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \]
\[ = 1 - 0 - 0 + 1 \]
\[ = 2 \]

Thus, \( \| u - v \| = \sqrt{2} \).

6.7.25. \( 1, t, 3t^2 - 1 \)

6.7.26. \( 1, t, 3t^2 - 4 \)