4.1.13.  
   a) No. There are three vectors in \( \{ v_1, v_2, v_3 \} \).
   b) There are infinitely many vectors in \( \text{Span} \{ v_1, v_2, v_3 \} \).
   c) Yes, \( w = v_1 + v_2 \).

4.1.14. We can answer this systematically by row reducing \((v_1, v_2, v_3, w)\).

\[
\begin{pmatrix}
1 & 2 & 4 & 8 \\
0 & 1 & 2 & 4 \\
-1 & 3 & 6 & 7 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

This tells us that \( w \) is not a linear combination of the \( v_i \). (We can also see, if somehow we had not yet noticed, that \( v_3 = 2v_2 \).

4.1.24.  
   a) True.
   b) True. (Blue box following vector space axioms, proof sketched in problem 29. Most importantly, this is not “by definition”.)
   c) Syntax error. It only makes sense to ask whether a vector space is a subspace of another vector space. Two true, well-formed sentences similar to the given one follow.
   
   • A vector space is always a subspace of itself.
   • A subspace of a vector space is always a vector space.
   d) False. \( \mathbb{R}^2 \) is not even a subset of \( \mathbb{R}^3 \).
   e) False. Quantifier error. We need \( u + v \in H \) for all \( u, v \in H \), and we need \( cu \in H \) for all \( c \in \mathbb{R}, u \in H \).

4.1.32. We can check the three parts of the definition directly.

   a) \( 0 \in H \) and \( 0 \in K \), so \( 0 \in H \cap K \)
   
   b) Suppose \( u, v \in H \cap K \). Then, since \( H \) is closed under addition, \( u + v \in H \). Since \( K \) is closed under addition, \( u + v \in K \). Thus \( u + v \in H \cap K \).
   
   c) Suppose \( u \in H \cap K \) and \( c \in \mathbb{R} \). Since \( H \) is closed under scalar multiplication, \( cu \in H \). Since \( K \) is closed under scalar multiplication, \( cu \in K \). Thus \( cu \in H \cap K \).

This shows that \( H \cap K \) is a subspace of \( V \).

Let \( H \) be the \( x \)-axis in \( \mathbb{R}^2 \) and let \( K \) be the \( y \)-axis, so that \( H \cup K \) looks like a cross. This is not closed under addition. For example, \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in H \cup K \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in H \cup K \), but \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin H \cup K \). (In general, the union of subspaces of \( V \) is almost never a subspace of \( V \). The only time \( H \cup K \) is a subspace of \( V \) is when one of \( H, K \) contains the other, so that \( H \cup K \) is just \( H \) or \( K \).)
4.1.33.

a) Since $0 \in H$ and $0 \in K$, $0 = 0 + 0 \in H + K$. Now, let $u = h + k$ and $v = h' + k'$ be arbitrary elements of $H + K$, and let $c \in \mathbb{R}$. Then $u + v = (h + h') + (k + k') \in H + K$, and also $cu = ch + ck \in H + K$.

b) We already know that $H$, $K$, and $H + K$ contain the zero vector in $V$ and are closed under linear combinations, so the only thing to check is that $H$ and $K$ are subsets of $H + K$. For all $h \in H$, $h = h + 0 \in H + K$. Likewise, for all $k \in K$, we have $k = 0 + k \in H + K$. Thus $H \subseteq H + K$ and $K \subseteq H + K$.

4.1.34. Let $u = h + k$ be any element of $H + K$. Then we can write $h = \sum_{i=1}^{p} c_i u_i$ and $k = \sum_{i=1}^{q} d_i v_i$ for suitable scalars $c_i, d_i$. Then we have

$$u = h + k = \sum_{i=1}^{p} c_i u_i + \sum_{i=1}^{q} d_i v_i \in \text{Span}\{u_1, \ldots, u_p, v_1, \ldots, v_q\},$$

which gives $H + K \subseteq \text{Span}\{u_1, \ldots, u_p, v_1, \ldots, v_q\}$.

For the other direction, let $u$ be any element of $\text{Span}\{u_1, \ldots, u_p, v_1, \ldots, v_q\}$. Then we can write

$$u = \sum_{i=1}^{p} c_i u_i + \sum_{i=1}^{q} d_i v_i = \left( \sum_{i=1}^{p} c_i u_i \right) + \left( \sum_{i=1}^{q} d_i v_i \right) \in H + K,$$

so that $\text{Span}\{u_1, \ldots, u_p, v_1, \ldots, v_q\} \subseteq H + K$.

Together, these given $H + K = \text{Span}\{u_1, \ldots, u_p, v_1, \ldots, v_q\}$.